Exercise 5.1
(Strong convergence)

Show that in Theorem 5.2.1, the convergence rate is of order 1 if σ is constant and b is $\mathcal{C}^2$ in space and $\mathcal{C}^1$ in time.

Solution: By Proposition 5.1.2, we can write the Euler solution as an Itô process

$$X_t^{(h)} = x + \int_0^t b(\varphi_s, X_{\varphi_s}^{(h)})\,\mathrm{d}s + \int_0^t \sigma(\varphi_s, X_{\varphi_s}^{(h)})\,\mathrm{d}W_s.$$

We can thus decompose the error as

$$\begin{aligned} E_t^{(h)} := X_t^{(h)} - X_t &= \int_0^t b(\varphi_s, X_{\varphi_s}^{(h)}) - b(s, X_s)\,\mathrm{d}s \\ &= \int_0^t b(\varphi_s, X_{\varphi_s}^{(h)}) - b(s, X_s^{(h)}) + b(s, X_s^{(h)}) - b(s, X_s)\,\mathrm{d}s, \end{aligned}$$

where the initial value x and volatility σ vanish since they are constant. Due to the continuity assumptions on b, we can apply the mean value theorem to the second part and bound the derivative, i.e.

$$b(s, X_s^{(h)}) - b(s, X_s) = b'_s(X_s^{(h)} - X_s) = b'_s E_s^{(h)} \leq \underbrace{\big(\max_{s\in[0,t]} b'_s\big)}_{=C} E_s^{(h)}.$$

Hence, we have that

$$E_t^{(h)} \leq \int_0^t b(\varphi_s, X_{\varphi_s}^{(h)}) - b(s, X_s^{(h)})\,\mathrm{d}s + C\int_0^t E_s^{(h)}\,\mathrm{d}s.$$

We seek to bound the L^p-norm of the error. Thus, we apply Minkowski's inequality to get

$$\begin{aligned} \|E_t^{(h)}\|_p &\leq \left\| \int_0^t b(\varphi_s, X_{\varphi_s}^{(h)}) - b(s, X_s^{(h)})\,\mathrm{d}s \right\|_p + C\left\|\int_0^t E_s^{(h)}\,\mathrm{d}s\right\|_p \\ &\leq \Bigg\| \underbrace{\int_0^t b(\varphi_s, X_{\varphi_s}^{(h)}) - b(s, X_s^{(h)})\,\mathrm{d}s}_{=:\alpha(t)} \Bigg\|_p + C\int_0^t \left\|E_s^{(h)}\right\|_p \mathrm{d}s. \end{aligned}$$

The goal here is to apply Gronwall's lemma and be done, but it requires first that $\|\alpha(t)\|_p \leq Ch$, since then the lemma gives us

$$\|E_t^{(h)}\|_p \leq Ch,$$

where C comes from the integral in the lemma. Now we note that applying the Lipschitz continuity of b, and inserting the definition of the Euler scheme, we get

$$
\begin{aligned}
|\alpha(t)| &= \int_0^t |b(\varphi_s, X_{\varphi_s}^{(h)}) - b(s, X_s^{(h)})| \, \mathrm{d}s \\
&\leq C_{b,\sigma} \int_0^t \underbrace{|s-\varphi_s|}_{\leq h} + |X_{\varphi_s}^{(h)} - X_s^{(h)}| \, \mathrm{d}s \\
&\leq C_{b,\sigma} T h + C_{b,\sigma} \int_0^t |X_{\varphi_s}^{(h)} - X_s^{(h)}| \, \mathrm{d}s \\
&= Ch + C \int_0^t |b(\varphi_s, X_{\varphi_s}^{(h)})(s-\varphi_s) + \sigma(\varphi_s, X_{\varphi_s}^{(h)})(W_s - W_{\varphi_s})| \, \mathrm{d}s \\
&\leq Ch + C b_{\max} \int_0^t \underbrace{|s-\varphi_s|}_{\leq h} \, \mathrm{d}s + C\sigma \int_0^t |W_s - W_{\varphi_s}| \, \mathrm{d}s \\
&\leq Ch + C \int_0^t |W_s - W_{\varphi_s}| \, \mathrm{d}s.
\end{aligned}
$$

Hence, the norm of $\alpha(t)$ becomes (by applying a generalization of the convexity inequality)

$$
\begin{aligned}
\|\alpha(t)\|_p^p &\leq \mathbb{E}\Big(\Big(Ch + C\int_0^t |W_s - W_{\varphi_s}| \, \mathrm{d}s\Big)^p\Big) \\
&\leq \mathbb{E}\Big(2^{p-1}\Big(Ch^p + \Big(C\int_0^t |W_s - W_{\varphi_s}| \, \mathrm{d}s\Big)^p\Big)\Big) \\
&\leq Ch^p + C\mathbb{E}\Big(\Big(\int_0^t |W_s - W_{\varphi_s}| \, \mathrm{d}s\Big)^p\Big).
\end{aligned}
$$

We are now done if the second term can be bounded by Ch^p. This is seen by

$$
\begin{aligned}
&\mathbb{E}\Big(\Big(\int_0^t |W_s - W_{\varphi_s}| \, \mathrm{d}s\Big)^p\Big) \\
&\qquad = \mathbb{E}\Big(\Big(\int_0^t \operatorname{sgn}\{W_s - W_{\varphi_s}\}(W_s - W_{\varphi_s}) \, \mathrm{d}s\Big)^p\Big) \\
&\qquad = \mathbb{E}\Big(\Big(\int_0^t \int_{\varphi_s}^s \operatorname{sgn}\{W_s - W_{\varphi_s}\} \, \mathrm{d}W_r \, \mathrm{d}s\Big)^p\Big) \\
&\qquad = \mathbb{E}\Big(\Big(\int_0^t \int_0^t \operatorname{sgn}\{W_s - W_{\varphi_s}\} \mathbb{1}_{\{\varphi_s, s\}} \, \mathrm{d}W_r \, \mathrm{d}s\Big)^p\Big) \\
&\qquad = \mathbb{E}\Big(\Big(\int_0^t \int_0^t \operatorname{sgn}\{W_s - W_{\varphi_s}\} \mathbb{1}_{\{\varphi_s, s\}} \, \mathrm{d}s \, \mathrm{d}W_r\Big)^p\Big) \\
&\qquad = \mathbb{E}\Big(\Big(\int_0^t \Big|\int_0^t \operatorname{sgn}\{W_s - W_{\varphi_s}\} \mathbb{1}_{\{\varphi_s, s\}} \, \mathrm{d}s\Big|^2 \, \mathrm{d}r\Big)^{p/2}\Big) \\
&\qquad = \mathbb{E}\Big(\Big(\int_0^t |\underbrace{s-\varphi_s}_{\leq h}|^2 \, \mathrm{d}r\Big)^{p/2}\Big) = (Ch^2)^{p/2} = Ch^p.
\end{aligned}
$$