Exercise 5.2

(Milstein scheme)

Denote by $(X_t)_{t\geq 0}$ the solution of the stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^t b(X_s)\,\mathrm{d}s,$$

where $\sigma, b : \mathbb{R} \to \mathbb{R}$ are bounded $\mathcal{C}^2$-functions with bounded derivatives.

1. Show the short time L_2-approximation

$$\mathbb{E}\Big(\big(X_t - [x + b(x)t + \sigma(x)W_t]\big)^2\Big) = \frac{(\sigma\sigma'(x))^2}{2}t^2 + o(t^2).$$

<u>Solution:</u> First of all, there is a question on how to interpret $o(\cdot)$ here. In general, it is defined as $f(x) \in o(g(x))$ if it holds that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

However, what makes sense here is that we are supposed to interpret t as the size of an interval in the Milstein scheme, so we make the assumption that we consider the case when $t \to 0$, and proceed the exercise given this.

We apply Itô's formula on b and σ and integrate from 0 to t and note that

$$\begin{aligned}\int_0^t b(X_s)\,\mathrm{d}s &= b(x)t + \int_0^t \int_0^s b'(X_r)b(X_r) + \frac{1}{2}b''(X_r)\sigma(X_r)^2\,\mathrm{d}r\,\mathrm{d}s \\ &\quad + \int_0^t \int_0^s b'(X_r)\sigma(X_r)\,\mathrm{d}W_r\,\mathrm{d}s, \\ \int_0^t \sigma(X_s)\,\mathrm{d}W_s &= \sigma(x)W_t + \int_0^t \int_0^s \sigma'(X_r)b(X_r) + \frac{1}{2}\sigma''(X_r)\sigma(X_r)^2\,\mathrm{d}r\,\mathrm{d}W_s \\ &\quad + \int_0^t \int_0^s \sigma'(X_r)\sigma(X_r)\,\mathrm{d}W_r\,\mathrm{d}W_s.\end{aligned}$$

At this point, the integrals with $\mathrm{d}r\mathrm{d}s$ and $\mathrm{d}W_s\mathrm{d}r$ can be neglected, and we only need to consider the integrals with $\mathrm{d}W_s\mathrm{d}W_r$ (since the other integrals grow faster and will hence be included in the same $o(\cdot)$). For example, looking at the first integral in the expression for b, we get

$$\mathbb{E}\left(\left(\int_0^t \int_0^s (\text{bounded integrand})\,\mathrm{d}r\,\mathrm{d}s\right)^2\right) \sim Ct^4.$$

Hence, the expectation we seek will be equivalent to

$$\mathbb{E}\left(\left(\int_0^t \int_0^s \sigma'(X_r)\sigma(X_r)\,\mathrm{d}W_r\,\mathrm{d}W_s\right)^2\right) = \int_0^t \int_0^s \mathbb{E}\left((\sigma'(X_r)\sigma(X_r))^2\right)\mathrm{d}r\mathrm{d}s,$$

where the Itô isometry has been applied twice. Next, consider Itô's formula on the function $f(x) = x^2$ with $x = \sigma'(x)\sigma(x)$, and we find

$$(\sigma'(X_t)\sigma(X_t))^2 = (\sigma'(x)\sigma(x))^2 + \int_0^t 2\sigma'(X_s)\sigma(X_s)b(X_s) + \frac{1}{2}2\sigma(X_s)^2 \,\mathrm{d}s + \int_0^t 2\sigma'(X_s)\sigma(X_s)\sigma(X_s)\,\mathrm{d}W_s.$$

Here, the first constant stay as it is, the second term will in total join the $o(t^2)$ (since it itself is $o(t)$ and inserted in the double integral will become of order t^3, hence $o(t^2)$), and the last integral is the expectation of a stochastic integral (thus centered) and is evaluated to 0. Thus we find that

$$\begin{aligned}
\mathbb{E}\left((X_t - [x + b(x)t + \sigma(x)W_t])^2\right) &= \int_0^t \int_0^s \mathbb{E}\left((\sigma'(X_r)\sigma(X_r))^2\right) \mathrm{d}r\mathrm{d}s + o(t^2) \\
&= \frac{(\sigma'(x)\sigma(x))^2}{2} t^2 + o(t^2).
\end{aligned}$$

2. Similarly, show

$$\mathbb{E}\left(\left(X_t - [x + b(x)t + \sigma(x)W_t + \frac{1}{2}\sigma\sigma'(x)(W_t^2 - t)]\right)^2\right) = \mathcal{O}(t^3).$$

<u>Solution:</u> For this task, we follow the procedure of the previous exercise (i.e. applying Itô's formula on $b(x)$ and $\sigma(x)$), but note that

$$\int_0^t \int_0^s \sigma'(X_r)\sigma(X_r)\,\mathrm{d}W_r\mathrm{d}W_s = \int_0^t \int_0^s \left(\sigma'(x)\sigma(x) + \int_0^r b(X_y)\,\mathrm{d}y + \int_0^r \sigma(X_y)\,\mathrm{d}W_y\right) \mathrm{d}W_r\mathrm{d}W_s,$$

where we have simply written $\sigma'(X_r)\sigma(X_r)$ as its expression as an Itô process. Here it suffices to note that

$$\int_0^t \int_0^s \sigma'(x)\sigma(x)\,\mathrm{d}W_r\mathrm{d}W_s = \sigma'(x)\sigma(x)\int_0^t W_s\,\mathrm{d}W_s = \frac{1}{2}\sigma'(x)\sigma(x)(W_t^2 - t),$$

and that the expectation squared of the other integrals become as

$$\mathbb{E}\left(\int_0^t \int_0^s \int_0^r b(X_y)\,\mathrm{d}y\mathrm{d}W_r\mathrm{d}W_s\right)^2 = \mathbb{E}\int_0^t \int_0^s \Bigg(\underbrace{\int_0^r b(X_y)\,\mathrm{d}y}_{=\mathcal{O}(t)}\Bigg)^2 \mathrm{d}r\mathrm{d}s = \mathcal{O}(t^4),$$

$$\mathbb{E}\left(\int_0^t \int_0^s \int_0^r \sigma(X_y)\,\mathrm{d}W_y\mathrm{d}W_r\mathrm{d}W_s\right)^2 = \mathbb{E}\int_0^t \int_0^s \int_0^r \sigma(X_y)^2\,\mathrm{d}y\mathrm{d}r\mathrm{d}s = \mathcal{O}(t^3).$$

Hence we find that

$$\mathbb{E}\left(\left(X_t - [x + b(x)t + \sigma(x)W_t + \frac{1}{2}\sigma\sigma'(x)(W_t^2 - t)]\right)^2\right) = \mathcal{O}(t^3),$$

since the remaining integrals that appear from the Itô formula on $b(x)$ and $\sigma(x)$ can be found to be of $\mathcal{O}(t^3)$ when squaring them and taking the expectation. The reason why it suffices to check the expectation of the square of each integral for itself is that we can simply apply the convexity inequality on the expression.

3. The estimate in previous task leads to a high-order scheme, called the Milstein scheme, which is written

$$
\begin{aligned}
X_0^{(h,M)} &= x, \\
X_{(i+1)h}^{h,M} &= X_{ih}^{(h,M)} + b(X_{ih}^{(h,M)})h + \sigma(X_{ih}^{(h,M)})(W_{(i+1)h} - W_{ih}) \\
&\qquad\qquad + \frac{1}{2}\sigma\sigma'(X_{ih}^{(h,M)})[(W_{(i+1)h} - W_{ih})^2 - h].
\end{aligned}
$$

Use the estimate derived in previous task to prove that

$$\sup_{0\le i\le N} \mathbb{E}(|X_{ih}^{(h,M)} - X_{ih}|^2) = \mathcal{O}(h^2).$$

<u>Solution:</u> We split the error as

$$X_{nh} - X_{nh}^{(h,M)} = X_{nh} - X_{(n-1)h} + X_{nh}^{(h,M)} - X_{(n-1)h}^{(h,M)} + X_{(n-1)h} - X_{(n-1)h}^{(h,M)}, \tag{1}$$

and start by analyzing the contribution from the last part, i.e., $X_{(n-1)h} - X_{(n-1)h}^{(h,M)}$. This is done by proof of induction. That is, we begin by assuming that

$$\mathbb{E}\{(X_{ih} - X_{ih}^{(h,M)})^2\} = \mathcal{O}(h^3)$$

holds for i. For $i+1$, we then have

$$
\begin{aligned}
&\mathbb{E}\{(X_{(i+1)h} - X_{(i+1)h}^{(h,M)})^2\} \\
&= \mathbb{E}\{X_{(i+1)h} - (X_{ih} + b(X_{ih})h + \sigma(X_{ih})\Delta W_{(i+1)h} + \frac{1}{2}\sigma\sigma'(X_{ih})(\Delta W_{(i+1)h}^2 - h) \\
&\qquad + (X_{ih} + b(X_{ih})h + \sigma(X_{ih})\Delta W_{(i+1)h} + \frac{1}{2}\sigma\sigma'(X_{ih})(\Delta W_{(i+1)h}^2 - h) \\
&\qquad - (X_{ih}^{(h,M)} + b(X_{ih}^{(h,M)})h + \sigma(X_{ih}^{(h,M)})\Delta W_{(i+1)h} + \frac{1}{2}\sigma\sigma'(X_{ih}^{(h,M)})(\Delta W_{(i+1)h}^2 - h)))^2\} \\
&\le 2Ch^3 + 2\cdot 4\mathbb{E}\{(X_{ih} - X_{ih}^{(h,M)})^2\}(1 + \mathrm{Lip}_b^2 h^2 + \mathrm{Lip}_\sigma^2 h + \frac{1}{4}\mathrm{Lip}_{\sigma\sigma'}^2 h^2) = \mathcal{O}(h^3).
\end{aligned}
$$

Here, we first added and subtracted the two center terms, followed by the convex inequality to separate the two parts. We then used the estimate derived in previous exercise for the first part, and finally used the Lipschitz continuity of b and σ for the second part.

For the contribution of the first two terms in (1), we have

$$
\begin{aligned}
&\mathbb{E}\{((X_{(i+1)h} - X_{ih}) + (X^{(h,M)}_{(i+1)h} - X^{(h,M)}_{ih}))^2\} \\
&\quad = \mathbb{E}\{(X_{(i+1)h} - X_{ih} - (b(X_{ih})h + \sigma(X_{ih})\Delta W_{(i+1)h} + \frac{1}{2}\sigma\sigma'(X_{ih})(\Delta W^2_{(i+1)h} - h)) \\
&\qquad\qquad + (b(X_{ih})h + \sigma(X_{ih})\Delta W_{(i+1)h} + \frac{1}{2}\sigma\sigma'(X_{ih})(\Delta W^2_{(i+1)h} - h)) \\
&\qquad\qquad - (b(X^{(h,M)}_{ih})h + \sigma(X^{(h,M)}_{ih})\Delta W_{(i+1)h} + \frac{1}{2}(\Delta W^2_{(i+1)h} - h)))^2\} \\
&\quad \le \ 2Ch^3 + 2h[\mathrm{Lip}_b^2 h + \mathrm{Lip}_\sigma^2 + \frac{1}{4}\mathrm{Lip}^2_{\sigma\sigma'} h] \underbrace{\mathbb{E}\{(X_{ih} - X^{(h,M)}_{ih})^2\}}_{\le C(ih)\cdot h^2} \le 2Ch^3 + 2Ch^3(ih).
\end{aligned}
$$

Above, we first added and subtracted the two middle terms, used earlier results, and the Lipschitz continuity of b and σ. In total, we have

$$
\begin{aligned}
&\sup_{n \le i+1} \mathbb{E}\{(X_{nh} - X^{(h,M)}_{nh})^2\} \\
&\qquad \le \sup_{n \le i} 2 \cdot \mathbb{E}\{(X_{ih} - X^{(h,M)}_{ih})^2\} + 2 \cdot \mathbb{E}\{(X_{(i+1)h} - X_{ih} + (X^{(h,M)}_{(i+1)h} - X^{(h,M)}_{ih}))^2\} \\
&\qquad \le 2C \cdot ih \cdot h^2 + 2Ch^3 + 2Ch^2 \cdot ih = \mathcal{O}(h^2),
\end{aligned}
$$

since we can bound $ih \le T$ for a finite time T.