Exercise 5.3

(Convergence rate of weak convergence)

We consider the model of geometric Brownian motion:

$$X_t = x + \int_0^t \sigma X_s \, \mathrm{d}W_s + \int_0^t \mu X_s \, \mathrm{d}s,$$

with $x > 0$.

1. Compute $\mathbb{E}(X_T^2)$.

Solution: We start by finding the exact solution for the equation. Apply Itô's formula on $f(x) = \ln(x)$ and note that

$$\begin{aligned}\ln(X_t) &= \ln(X_0) + \int_0^t \frac{1}{X_s}\mu X_s - \frac{1}{2X_s^2}\sigma^2 X_s^2 \, \mathrm{d}s + \int_0^t \frac{1}{X_s}\sigma X_s \, \mathrm{d}W_s \\ &= \ln(x) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t,\end{aligned}$$

and thus $X_T = xe^{(\mu-\sigma^2/2)T+\sigma W_T}$. Squaring this, we note that

$$X_T^2 = x^2 e^{2\mu T - \sigma^2 T + 2\sigma W_T}.$$

Now, we use the fact that for a random variable $Z \sim \mathcal{N}(0, \xi^2)$ it holds that $\mathbb{E}(e^{cZ}) = e^{\frac{1}{2}c^2\sigma^2}$. Since $\mathrm{Var}(W_T) = T$, we get

$$\begin{aligned}\mathbb{E}(X_T^2) &= \mathbb{E}(x^2 e^{2\mu T - \sigma^2 T + 2\sigma W_T}) = x^2 e^{2\mu T - \sigma^2 T}\mathbb{E}(e^{2\sigma W_T}) \\ &= x^2 e^{2\mu T - \sigma^2 T} e^{2\sigma^2 T} = x^2 e^{2\mu T + \sigma^2 T}.\end{aligned}$$

2. Let $X^{(h)}$ be the related Euler scheme with time step h. Set $y_i = \mathbb{E}((X_{ih}^{(h)})^2)$. Find a relation between y_{i+1} and y_i.

Solution: We have that (by looking at the Euler scheme for $X_{(i+1)h}^{(h)}$ and moving the first term to the left hand side)

$$\begin{aligned}\mathbb{E}\big((X_{(i+1)h}^{(h)} - X_{ih}^{(h)})^2\big) &= \mathbb{E}\big((b(X_{ih}^{(h)})h + \sigma(X_{ih}^{(h)})(W_{(i+1)h} - W_{ih}))^2\big) \\ &= \mathbb{E}\big((\mu X_{ih}^{(h)} h + \sigma X_{ih}^{(h)}(W_{(i+1)h} - W_{ih}))^2\big) \\ &= \mathbb{E}\big(\mu^2 \big(X_{ih}^{(h)}\big)^2 h^2 + \sigma^2 \big(X_{ih}^{(h)}\big)^2 (W_{(i+1)h} - W_{ih})^2)\big) \\ &= \mathbb{E}\big(\big(X_{ih}^{(h)}\big)^2\big)(\mu^2 h^2 + \sigma^2 h) \\ &= y_i(\mu^2 h^2 + \sigma^2 h),\end{aligned}$$

where in the third step the mixed term vanishes since the expected value of the Brownian increment is 0. Next we note that

$$\begin{aligned}\mathbb{E}\big(X_{ih}^{(h)}\big(X_{(i+1)h}^{(h)} - X_{ih}^{(h)}\big)\big) &= \mathbb{E}\big(X_{ih}^{(h)}\big(\mu X_{ih}^{(h)} h + \sigma X_{ih}^{(h)}(W_{(i+1)h} - W_{ih})\big)\big) \\ &= \mathbb{E}\big(\big(X_{ih}^{(h)}\big)^2\big)\mu h = y_i \mu h,\end{aligned}$$

where the second term vanished, once again, due to expectation 0 from the increments. Now, we can find a relation by writing

$$\begin{aligned}
y_{i+1} &= \mathbb{E}\Big(\Big(X^{(h)}_{(i+1)h}\Big)^2\Big) = \mathbb{E}\Big(\Big(X^{(h)}_{(i+1)h} - X^{(h)}_{ih} + X^{(h)}_{ih}\Big)^2\Big) \\
&= \mathbb{E}\Big(\big(X^{(h)}_{(i+1)h} - X^{(h)}_{ih}\big)^2 + 2X^{(h)}_{ih}\big(X^{(h)}_{(i+1)h} - X^{(h)}_{ih}\big) + \big(X^{(h)}_{ih}\big)^2\Big) \\
&= y_i(\mu^2 h^2 + \sigma^2 h) + 2y_i\mu h + y_i \\
&= y_i(1 + 2\mu h + \sigma^2 h + \mu^2 h^2).
\end{aligned}$$

3. Deduce that $\mathbb{E}((X^{(h)}_T)^2) = \mathbb{E}(X_T^2) + \mathcal{O}(h)$.

Solution: From previous exercise, we deduce

$$y_{i+1} = (1 + (2\mu + \sigma^2 + \mu^2 h)h)y_i = (1 + (2\mu + \sigma^2 + \mu^2 h)h)^{i+1}x^2$$

since $y_0 = x^2$. Moreover, if we set the number of time steps to N, we have $h = T/N$, and thus

$$\begin{aligned}
\mathbb{E}\{(X^{(h)}_T)^2\} = y_N &= \left(1 + \frac{(2\mu + \sigma^2 + \mu^2 h)T}{N}\right)^N x^2 \\
&\le e^{(2\mu+\sigma^2+\mu^2 h)T}x^2 \\
&= e^{(2\mu+\sigma^2+\mu^2 h)T}x^2 + x^2 e^{2\mu T + \sigma^2 T} - x^2 e^{2\mu T + \sigma^2 T} \\
&= \mathbb{E}\{X_T^2\}\} + x^2\left(e^{(2\mu+\sigma^2+\mu^2 h)T} - e^{2\mu T+\sigma^2 T}\right) \\
&= \mathbb{E}\{X_T^2\}\} + x^2\left(e^{2\mu T+\sigma^2 T}(e^{\mu^2 hT} - 1)\right) \\
&\le E\{X_T^2\} + \mathcal{O}(h),
\end{aligned}$$

which concludes the exercise. Above we began by using the fact that $\lim_{n\to\infty}(1 + x/n)^n = e^x$ is a monotone sequence. Further, we added and subtracted $\mathbb{E}\{X_T^2\}$, and in the end we utilized Taylor's formula to conclude that $e^{\mu^2 hT} - 1 = \mu^2 hT + \mathcal{O}(h^2)$, which in total gave us the $\mathcal{O}(h)$-term in the end.