MVE565/MMA630
Exercise sessions

This document contains the exercises that are to be presented on the exercise sessions in the course Computational Methods for Stochastic Differential Equations (MVE565/MMA630). The exercises are taken directly from the course literature *"Monte-Carlo Methods and Stochastic Processes: From Linear to Non-Linear"* by Emmanuel Gobet. This document has further provided hints to some of the exercises. However, it should be clarified that several of the exercises are not restricted to a unique solution, and it is not required to follow given hints.

By solving the exercises, there is a possibility to gain bonus points for the final exam. For each exercise (or sub-exercise) that is solved, one gains a credit, and a certain amount of credits will in turn yield bonus points. To obtain the credits, one must be ready to present the exercises on the blackboard on the exercise sessions. By the end of the course, 50% of the credits will grant 1 bonus point for the final exam, and 80% will grant 2 bonus points.

Exercise session 1

Exercise 4.1 (Linear transformation of Brownian motion) (1 credit)

i) Let W be a standard d-dimensional Brownian motion and let U be an orthogonal matrix (i.e. $U^T = U^{-1}$). Prove that UW defines a new standard d-dimensional Brownian motion.

ii) Let W_1 and W_2 be two independent Brownian motions. For any $\rho \in [-1, 1]$, justify that $\rho W_1 + \sqrt{1-\rho^2} W_2$ and $-\sqrt{1-\rho^2} W_1 + \rho W_2$ are two independent Brownian motions.

Hints: None.

Exercise 4.2 (Approximation of the integral of a stochastic process)

For a standard Brownian motion, we study the convergence rate of the approximation

$$\Delta I_n := \int_0^1 W_s \, \mathrm{d}s - \frac{1}{n} \sum_{i=0}^{n-1} W_{\frac{i}{n}},$$

as $n \to +\infty$.

i) (Rough estimate). Prove that

$$\mathbb{E}(|\Delta I_n|) \leq \sum_{i=0}^{n-1} \mathbb{E} \left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} \left| W_s - W_{\frac{i}{n}} \right| \mathrm{d}s \right) = \mathcal{O}(n^{-1/2}). \qquad \textbf{(1 credit)}$$

Hints: For the inequality, one can start by dividing the integral $\int_0^1 W_s \, \mathrm{d}s$ into a sum of n integrals, partitioned on $[0, 1]$, and continue from there. For the last equality, one can apply Fubini's theorem and calculate $\mathbb{E}(|X|)$ for $X \sim \mathcal{N}(0, \sigma^2)$.

ii) Using Lemma A.1.4, prove that ΔI_n is Gaussian distributed. Compute its parameters and conclude that

$$\mathbb{E}(|\Delta I_n|) = \mathcal{O}(n^{-1}). \qquad \textbf{(1 credit)}$$

Hints: One can write ΔI_n as a sum of n integrals (as in the hint in previous task). For one of these integrals, one can view it as the limit of a Riemann sum, called S_M. Show S_M is Gaussian and compute its expectation and variance, and apply Lemma A.1.4 by taking corresponding limits as $M \to \infty$. The last part can then be computed since you have shown ΔI_n is Gaussian and you know its expectation and variance.

iii) A more generic proof of the above estimate consists of writing

$$\Delta I_n := \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(\frac{i+1}{n} - s \right) \mathrm{d}W_s.$$

Show this by applying the Itô formula to $s \mapsto (\frac{i+1}{n} - s)(W_s - W_{\frac{i}{n}})$ on each interval $[\frac{i}{n}, \frac{i+1}{n}]$. Using the Itô isometry, derive $\mathbb{E}(|\Delta I_n|^2) = \mathcal{O}(n^{-2})$ and therefore the announced estimate. **(1 credit)**

Hints: None.

iv) Proceeding as in *(iii)*, extend the previous estimate to

$$\Delta I'_n := \int_0^1 X_s \, \mathrm{d}s - \frac{1}{n} \sum_{i=0}^{n-1} X_{\frac{i}{n}},$$

where X is a scalar Itô process with bounded coefficients. **(1 credit)**

Hints: As earlier, one can write $\Delta I'_n$ as a sum of n integrals. Apply Itô's formula using $f(s, X_s) = (\frac{i+1}{n} - s)(X_s - X_{\frac{i}{n}})$, and bound $\Delta I'_n$ using the fact that the coefficients are bounded. Use this bound to further show the sought estimate $\mathbb{E}(|\Delta I'_n|^2) = \mathcal{O}(n^{-2})$. When showing this, you can use the fact that $\mathbb{E}(|\Delta I_n|^2) = \frac{1}{3n^2}$.

Exercise session 2

Exercise 4.3 (Approximation of stochastic integral)

We consider the convergence rate of the approximation

$$\Delta J_n := \int_0^1 Z_s \,\mathrm{d}W_s - \sum_{i=0}^{n-1} Z_{\frac{i}{n}}(W_{\frac{i+1}{n}} - W_{\frac{i}{n}})$$

where $Z_s := f(s, W_s)$ for some function f, such that $\mathbb{E}\int_0^1 |Z_s|^2 \,\mathrm{d}s + \sup_{i<n} \mathbb{E}(|Z_{\frac{i}{n}}|^2) < +\infty$. We illustrate that the convergence order is, under mild conditions, equal to $1/2$ but it can be smaller for irregular f.

i) Show that

$$\mathbb{E}(|\Delta J_n|^2) = \mathbb{E}\bigg(\sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} |Z_s - Z_{\frac{i}{n}}|^2 \,\mathrm{d}s\bigg). \qquad \textbf{(1 credit)}$$

Hints: Note that $\int_{i/n}^{(i+1)/n} \mathrm{d}W_s = W_{\frac{i+1}{n}} - W_{\frac{i}{n}}$, so the terms in the sum of ΔJ_n can be written as stochastic integrals. One may then write ΔJ_n as one integral and apply Itô isometry.

ii) When $Z_s = W_s$, show that $\mathbb{E}(|\Delta J_n|^2) \sim \mathrm{Cst}\, n^{-1}$ for some positive constant. **(1 credit)**

Hints: Start by using the result from *(i)*.

iii) Assuming that f is bounded, smooth with bounded derivatives, prove that $\mathbb{E}(|\Delta J_n|^2) = \mathcal{O}(n^{-1})$. **(1 credit)**

Hints: Bound $Z_s - Z_{\frac{i}{n}}$ by using Itô's formula. Use the result from *(i)* in combination with the derived bound.

iv) Assume that Z is a square-integrable martingale. Show that $\mathbb{E}(|Z_s - Z_{\frac{i}{n}}|^2) \le \mathbb{E}(|Z_{\frac{i+1}{n}}|^2) - \mathbb{E}(|Z_{\frac{i}{n}}|^2)$, and thus $\mathbb{E}(|\Delta J_n|^2) \le (\mathbb{E}(|Z_1|^2) - \mathbb{E}(|Z_0|^2)n^{-1}$. **(1 credit)**

Hints: For the first part, use towering property to show that $\mathbb{E}(|Z_s - Z_{\frac{i}{n}}|^2) = \mathbb{E}(Z_s^2) - \mathbb{E}(Z_{\frac{i}{n}}^2)$. Then show that $\mathbb{E}(Z_s^2) \le \mathbb{E}(Z_{\frac{i+1}{n}}^2)$ to get the sought inequality. Use the derived inequality, in combination with the result from *(i)* to compute the last inequality.

Exercise 4.4 (Exact simulation of Ornstein–Uhlenbeck process)

Let us consider the Ornstein–Uhlenbeck process $(X_t)_{t\ge 0}$, the solution of

$$X_t = x_0 - a\int_0^t X_s \,\mathrm{d}s + \sigma W_t,$$

where $x_0 \in \mathbb{R}$, $\sigma > 0$, and $(W_t)_{t\ge0}$ is a standard Brownian motion.

i) By applying the Itô formula to $e^{at}X_t$, give an explicit representation for X_t in terms of stochastic integrals. **(1 credit)**

Hints: None.

ii) Deduce the explicit distribution of $(X_{t_1}, ..., X_{t_n})$. **(1 credit)**

Hints: The sought representation of X_t from previous task is given by $X_t = x_0 e^{-at} + \sigma \int_0^t e^{a(s-t)}\, \mathrm{d}W_s$. For the computation of the covariance, it is convenient to apply the covariance property (stated on page 136 in the course literature).

iii) Find two functions $\alpha(t)$ and $\beta(t)$ such that $(X_t)_{t\geq 0}$ has the same distribution as $(Y_t)_{t\geq 0}$ with $Y_t = \alpha(t)(x_0 + W_{\beta(t)})$. **(1 credit)**

Hints: The result in *(ii)* is that X_t is Gaussian with expectation $\mathbb{E}(X_t) = x_0 e^{-at}$ and covariance $\mathrm{Cov}(X_t X_s) = \frac{\sigma^2}{2a} e^{-a(t-s)}(1 - e^{-2as})$.

Exercise session 3

Exercise 4.5 (Transformations of SDE and PDE)

For any $t \in [0,T)$ and $x \in \mathbb{R}$, we denote by $(X_s^{t,x}, s \in [t,T])$ the solution to

$$X_s = x + \int_t^s b(X_r)\,\mathrm{d}r + \int_t^s \sigma(X_r)\,\mathrm{d}W_r, \qquad t \le s \le T,$$

where the coefficients $b, \sigma : \mathbb{R} \to \mathbb{R}$ are smooth with bounded derivatives, and $\sigma(x) \ge c > 0$. For a given Borel set $A \subset \mathbb{R}$ we define $u(t,x) := \mathbb{P}(X_T^{t,x} \in A)$. We assume in the following $u(t,x) > 0$ for any $(t,x) \in [0,T) \times \mathbb{R}$, and that appropriate smoothness assumptions are satisfied (namely, $u \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R})$).

i) Let $x_0 \in \mathbb{R}$ and f be a bounded continuous function. Using the PDE satisfied by u on $[0,T) \times \mathbb{R}$, show that

$$\mathbb{E}(f(X_t)|X_T \in A) = \frac{\mathbb{E}(f(X_t)u(t,X_t))}{u(0,x_0)}, \qquad \forall t < T,$$

where $X_t = X_t^{0,x_0}$ to simplify. **(1 credit)**

Hints: Use definition of conditional expectation, and apply towering property with a filtration $\mathcal{F}_t$.

ii) We assume that for any $s \le t < T$ the equation

$$\overline{X}_r = x + \int_s^r \left(b(\overline{X}_w) + \sigma^2(\overline{X}_w)\frac{\partial_x u}{u}(w, \overline{X}_w) \right) \mathrm{d}w + \int_s^r \sigma(\overline{X}_w)\,\mathrm{d}W_w, \quad s \le r \le t$$

has a unique solution, denoted by $(\overline{X}_r^{s,x}, s \le r \le t)$. We set $v_t(s,x) := \mathbb{E}\big(f(\overline{X}_t^{s,x})\big)$.

a) What is the PDE solved by $(s,x) \mapsto v_t(s,x)$ on $[0,t) \times \mathbb{R}$? **(1 credit)**

   Hints: None.

b) Applying the Itô formula to $u(s,X_s)$ and $v_t(s,x)$, $0 \le s \le t$, and then to $u(s,X_s)v_t(s,X_s)$, show

   $$\mathbb{E}(f(X_t)u(t,X_t)) = v_t(0,x_0)u(0,x_0), \quad \forall t < T. \qquad \textbf{(1 credit)}$$

   Hints: Apply Itô's lemma in the sense of product rule for Itô processes on $\mathrm{d}(v_t(s,X_s)u(s,X_s))$. Find an expression for $\mathrm{d}v_t$. For the $\mathrm{d}u$ term, one can first show $u(s,X_s)$ is a martingale, and then see what happens to the $\mathrm{d}u$ term by analyzing the Riemann sum. Integrate and take expectation of the expression for $\mathrm{d}(v_t(s,X_s)u(s,X_s))$.

c) Conclude that for any $t < T$, the distribution of X_t given $\{X_T \in A\}$ is the distribution of $\overline{X}_t^{0,x_0}$. **(1 credit)**

   Hints: None.

iii) We skip this one.

Exercise 5.1 (Strong convergence)

Show that in Theorem 5.2.1, the convergence rate is of order 1 if σ is constant and b is $\mathcal{C}^2$ in space and $\mathcal{C}^1$ in time.

We divide the exercise into two separate parts.

i) Show that the error $E_t^{(h)} := X_t^{(h)} - X_t$ can be bounded in L^p-norm as

$$\|E_t^{(h)}\|_p \leq \Bigg\| \underbrace{\int_0^t b(\varphi_s, X_{\varphi_s}^{(h)}) - b(s, X_s^{(h)}) \,\mathrm{d}s}_{=:\alpha(t)} \Bigg\|_p + C \int_0^t \big\|E_s^{(h)}\big\|_p \,\mathrm{d}s. \qquad \textbf{(1 credit)}$$

Hints: None.

ii) Show that $\|\alpha(t)\|_p \leq Ch$. **(1 credit)**.

Hints: One can show that $|\alpha(t)| \leq Ch + C\int_0^t |W_s - W_{\varphi_s}| \,\mathrm{d}s$ by using the Lipschitz continuity of b. To further bound $\|\alpha(t)\|_p^p$, one can use a generalization of the convexity inequality.

Exercise session 4

Exercise 5.2 (Milstein scheme)

Denote by $(X_t)_{t\geq 0}$ the solution of the stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^t b(X_s)\,\mathrm{d}s,$$

where $\sigma, b : \mathbb{R} \to \mathbb{R}$ are bounded $\mathcal{C}^2$-functions with bounded derivatives.

1. Show the short time L_2-approximation

$$\mathbb{E}\Big(\big(X_t - [x + b(x)t + \sigma(x)W_t]\big)^2\Big) = \frac{(\sigma\sigma'(x))^2}{2}t^2 + o(t^2). \qquad \textbf{(1 credit)}$$

Hints: Itô's formula is your friend here. Start by applying it to b and σ and work your way from there.

2. Similarly, show

$$\mathbb{E}\left(\left(X_t - [x + b(x)t + \sigma(x)W_t + \frac{1}{2}\sigma\sigma'(x)(W_t^2 - t)]\right)^2\right) = \mathcal{O}(t^3). \qquad \textbf{(1 credit)}$$

Hints: Start similarly with Itô's formula on b and σ. Then try to write $\sigma'\sigma$ as an Itô process and analyze the expectation of each term that comes with it.

3. The estimate in previous task leads to a high-order scheme, called the Milstein scheme, which is written

$$\begin{aligned} X_0^{(h,M)} &= x, \\ X_{(i+1)h}^{h,M} &= X_{ih}^{(h,M)} + b(X_{ih}^{(h,M)})h + \sigma(X_{ih}^{(h,M)})(W_{(i+1)h} - W_{ih}) \\ &\qquad + \frac{1}{2}\sigma\sigma'(X_{ih}^{(h,M)})[(W_{(i+1)h} - W_{ih})^2 - h]. \end{aligned}$$

Use the estimate derived in previous task to prove that

$$\sup_{0\leq i\leq N} \mathbb{E}(|X_{ih}^{(h,M)} - X_{ih}|^2) = \mathcal{O}(h^2). \qquad \textbf{(1 credit)}$$

Hints: One can split the error as $X_{nh} - X_{nh}^{(h,M)} = X_{nh} - X_{(n-1)h} + X_{nh}^{(h,M)} - X_{(n-1)h}^{(h,M)} + X_{(n-1)h} - X_{(n-1)h}^{(h,M)}$. The contribution from the last part can be shown to satisfy $\mathbb{E}\{(X_{ih} - X_{ih}^{(h,M)})^2\} = \mathcal{O}(h^3)$ by proof of induction. For the induction, as well as later for the remaining contribution of the error split, the Lipschitz continuity of b and σ can be of use.

Exercise 5.3 (Convergence rate of weak convergence)

We consider the model of geometric Brownian motion:

$$X_t = x + \int_0^t \sigma X_s\,\mathrm{d}W_s + \int_0^t \mu X_s\,\mathrm{d}s,$$

with $x > 0$.

1. Compute $\mathbb{E}(X_T^2)$. **(1 credit)**

Hints: Apply Itô's formula on $f(x) = \ln(x)$.

2. Let $X^{(h)}$ be the related Euler scheme with time step h. Set $y_i = \mathbb{E}((X_{ih}^{(h)})^2)$. Find a relation between y_{i+1} and y_i. **(1 credit)**

Hints: One way is to take $+X_{ih} - X_{ih}$ in y_{i+1}, and derive expressions for each term in terms of y_i.

3. Deduce that $\mathbb{E}((X_T^{(h)})^2) = \mathbb{E}(X_T^2) + \mathcal{O}(h)$. **(1 credit)**

Hints: Derive an expression for $\mathbb{E}\{(X_T^{(h)})^2\} = y_N$ in terms of $y_0 = x^2$. One can then add and subtract $\mathbb{E}\{X_T^2\}$, and show the last part satisfies $\mathcal{O}(h)$ convergence rate.

Exercise session 5

Exercise 6.1 (Central limit theorem for varying h and M)

We study the CLT-type convergence of

$$\text{Error}_{h,M} = \frac{1}{M}\sum_{m=1}^{M} \mathcal{E}(f,g,k,X^{(h,m)}) - \mathbb{E}(\mathcal{E}(f,g,k,X))$$

by varying both the number of simulation M and the time step h. We consider the asymptotics $M \to +\infty$ and $h \to 0$, with different regimes on Mh^2.

i) Assume $Mh^2 \to 0$, that f, g, k are bounded continuous functions, and that the weak error is of order 1 w.r.t. h. Show a central limit theorem on $\sqrt{M}\text{Error}_{h,M}$ with a limit equal to a centered Gaussian random variable with variance $\mathbb{V}\text{ar}(\mathcal{E}(f,g,k,X))$. **(1 credit)**

Hints: Decompose the error into stastical and discretization errors, and conclude which part is sufficient to consider. To deduce the variance (and the fact that it is a Gaussian random variable), one can use a characteristic function.

ii) Assume that $Mh^2 = \text{Cst} \neq 0$, and prove a central limit theorem but with a non-centered Gaussian random variable at the limit. For this, we assume that the weak error can be expanded at order 1 w.r.t. h. **(1 credit)**

Hints: None.

Exercise 6.2 (Multi-level method with various strong convergence order)

Assume that the strong convergence of the Euler scheme is of order 1 w.r.t. h, and the weak convergence order is still 1 w.r.t h.

i) By a similar analysis to that of Theorem 6.3.1, determine the optimal allocation of computational effort within the different levels (as a function of number of simulations). **(1 credit)**

Hints: Follow the calculations on page 200 in the course literature, but utilize the fact the we have higher order of convergence for the strong error. Find corresponding expressions for $\mathbb{E}\{\text{Error}^2_{h,M}\}$ and $\mathcal{C}_{\text{cost}}$, and deduce optimal choice of M_l. This can be done by solving the optimization problem to minimize $\mathcal{C}_{\text{cost}}$ subject to $\mathbb{E}\{\text{Error}^2_{h,M}\} \leq \varepsilon^2$, using, e.g., Lagrange multiplier method.

ii) What is the global complexity $\mathcal{C}_{\text{cost}}$ as a function of the tolerance error ε? **(1 credit)**

Hints: Adapt the calculations made on page 201 in the course literature.

iii) More generally, assume that the Euler scheme converges strongly at order $\alpha \in (0,1]$, and weakly of order $\beta \in (0,1]$ (observe that $\alpha \leq \beta$). Derive the complexity/accuracy analysis associated with a multi-level method. What are the configurations of (α, β) for which (after optimizing the effort within levels)

$$\mathcal{C}_{\text{cost}} \sim_c \varepsilon^{-2},$$

i.e. we retrieve the standard Monte-Carlo convergence rate? **(1 credit)**

Hints: One can make similar calculations as in *(i)*, but with new convergence orders α and β, and then optimize using Lagrange multipliers to find a more general result of the one in *(i)*.