[HRSW] Theorem 3.65 for Crank-Nicolson ($\theta = \frac{1}{2}$)

Proof: Need to obtain higher convergence when bounding

$$(r_1^m, v_N) = \left(k^{-1}(u^{m+1}-u^m) - \dot{u}^{m+\theta}, v_N\right)$$

$$= \left(k^{-1}\int_{t_m}^{t_{m+1}} \dot{u}(s)\,ds - \frac{1}{2}(\dot{u}^{m+1}+\dot{u}^m), v_N\right)$$

$$= \left(k^{-1}\frac{1}{2}\left(\int_{t_m}^{t_{m+1}} (\dot{u}(s)-\dot{u}^{m+1})\,ds + \int_{t_m}^{t_{m+1}} (\dot{u}(s)-\dot{u}^{m})\,ds\right), v_N\right)$$

$$= \left(k^{-1}\frac{1}{2}\left(-\int_{t_m}^{t_{m+1}}\int_{s}^{t_{m+1}} \ddot{u}(r)\,dr\,ds + \int_{t_m}^{t_{m+1}}\int_{t_m}^{s} \ddot{u}(r)\,dr\,ds\right), v_N\right)$$

$$= \Bigg(k^{-1}\frac{1}{2}\Bigg(-\int_{t_m}^{t_{m+1}}\int_{s}^{t_{m+1}} (\ddot{u}(r)-\ddot{u}(s))\,dr\,ds - \int_{t_m}^{t_{m+1}} (t_{m+1}-s)\ddot{u}(s)\,ds$$

$$-\int_{t_m}^{t_{m+1}}\int_{t_m}^{s} (\ddot{u}(s)-\ddot{u}(r))\,dr\,ds + \int_{t_m}^{t_{m+1}} (s-t_m)\ddot{u}(s)\,ds\Bigg), v_N\Bigg)$$

$$= \Bigg(k^{-1}\frac{1}{2}\Bigg(-\int_{t_m}^{t_{m+1}}\int_{s}^{t_{m+1}}\int_{s}^{r} \dddot{u}(t)\,dt\,dr\,ds - \int_{t_m}^{t_{m+1}}\int_{t_m}^{s}\int_{r}^{s} \dddot{u}(t)\,dt\,dr\,ds$$

$$-\int_{t_m}^{t_{m+1}} (t_{m+1}-2s+t_m)\ddot{u}(s)\,ds\Bigg), v_N\Bigg)$$

$$= \Bigg(k^{-1}\frac{1}{2}\Bigg(-\int_{t_m}^{t_{m+1}}\int_{t_m}^{t_{m+1}}\int_{t_m}^{t_{m+1}} \Big(\overbrace{\mathbb{1}_{[s,t_{m+1}]}(r)\,\mathbb{1}_{[s,r]}(t) + \mathbb{1}_{[t_m,s]}(r)\,\mathbb{1}_{[r,s]}(t)}^{=:\mathbb{1}(s,r,t)}\Big)\dddot{u}(t)\,dt\,dr\,ds$$

$$-\int_{t_m}^{t_{m+1}} (t_{m+1}-2s+t_m)(\ddot{u}(s)-\ddot{u}^m)\,ds - \underbrace{\int_{t_m}^{t_{m+1}} (t_{m+1}-2s+t_m)\ddot{u}^m\,ds}_{=0}\Bigg), v_N\Bigg)$$

$$= \left(k^{-1}\frac{1}{2}\left(-\int_{t_m}^{t_{m+1}}\int_{t_m}^{t_{m+1}}\int_{t_m}^{t_{m+1}} \mathbb{1}(s,r,t)\,\dddot{u}(t)\,dt\,dr\,ds - \int_{t_m}^{t_{m+1}} (t_{m+1}-2s+t_m)\int_{t_m}^{s}\dddot{u}(r)\,dr\,ds\right), v_N\right)$$

$$\Rightarrow |(r_1^m, v_N)| \leq \|r_1^m\|_* \, \|v_N\|_a$$

and

$$\|r_1^m\|_* \leq k^{-1}\frac{1}{2}\left(\int_{t_m}^{t_{m+1}}\int_{t_m}^{t_{m+1}}\int_{t_m}^{t_{m+1}} \mathbb{1}(s,r,t)\,\|\dddot{u}(t)\|_*\,dt\,dr\,ds + \int_{t_m}^{t_{m+1}} |t_{m+1}-2s+t_m|\int_{t_m}^{s}\|\dddot{u}(r)\|_*\,dr\,ds\right)$$

$$\leq k^{-1}\frac{1}{2}\Bigg(\int_{t_m}^{t_{m+1}}\int_{t_m}^{t_{m+1}}\Big(\int_{t_m}^{t_{m+1}} (\underbrace{\mathbb{1}(s,r,t)}_{\leq 1})^2\,dt\Big)^{\frac{1}{2}}\Big(\int_{t_m}^{t_{m+1}} \|\dddot{u}(t)\|_*^2\,dt\Big)^{\frac{1}{2}}\,dr\,ds$$

$$+\int_{t_m}^{t_{m+1}} |t_{m+1}-2s+t_m|\Big(\int_{t_m}^{s} 1^2\,dr\Big)^{\frac{1}{2}}\Big(\int_{t_m}^{s} \|\dddot{u}(r)\|_*^2\,dr\Big)^{\frac{1}{2}}\,ds\Bigg)$$

$$\leq k^{-1}\frac{1}{2}\left(k^{5/2} + \frac{1}{2}k^2\cdot k^{\frac{1}{2}}\right)\left(\int_{t_m}^{t_{m+1}} \|\dddot{u}(r)\|_*^2\,dr\right)^{\frac{1}{2}} \leq \underline{\frac{3}{4}k^{3/2}}\left(\int_{t_m}^{t_{m+1}} \|\dddot{u}(r)\|_*^2\,dr\right)^{\frac{1}{2}}$$