Exercise 6.1

(Central limit theorem for varying h and M)

We study the CLT-type convergence of

$$\text{Error}_{h,M} = \frac{1}{M}\sum_{m=1}^{M}\mathcal{E}(f,g,k,X^{(h,m)}) - \mathbb{E}(\mathcal{E}(f,g,k,X))$$

by varying both the number of simulation M and the time step h. We consider the asymptotics $M \to +\infty$ and $h \to 0$, with different regimes on Mh^2.

i) Assume $Mh^2 \to 0$, that f, g, k are bounded continuous functions, and that the weak error is of order 1 w.r.t. h. Show a central limit theorem on $\sqrt{M}\text{Error}_{h,M}$ with a limit equal to a centered Gaussian random variable with variance $\mathbb{V}\text{ar}(\mathcal{E}(f,g,k,X))$.

Solution: For convenient notation, denote $\mathcal{E}(X) := \mathcal{E}(f,g,k,X)$. We begin by decomposing the error as

$$\text{Error}_{h,M} = \underbrace{\frac{1}{M}\sum_{m=1}^{M}\mathcal{E}(X^{(h,m)}) - \mathbb{E}(\mathcal{E}(X^{(h)}))}_{=:S_M} + \underbrace{\mathbb{E}(\mathcal{E}(X^{(h)})) - \mathbb{E}(\mathcal{E}(X))}_{=:D_h},$$

where we have denoted the statistical error by S_M and the discretization error by D_h. Since $Mh^2 \to 0$, we have that $M \ll h^2$, and can hence neglect the discretization error and only analyze the statistical error S_M. The fact that it converges to a Gaussian random variable will be seen in the end when we analyze the limit of the characteristic function. For the expectation however, we note right away that

$$\mathbb{E}(\sqrt{M}\text{Error}_{h,M}) = \frac{1}{\sqrt{M}}\sum_{m=1}^{M}\mathbb{E}(\mathcal{E}(X^{(h,m)})) - \sqrt{M}\mathbb{E}(\mathcal{E}(X^{(h)})) = 0,$$

since the distribution of each sample $X^{(h,m)}$ is equal to that of $X^{(h)}$. To check the variance of the limit, we begin by assuming $\mathcal{E}(X) = 0$ (this is possible since we otherwise simply can recenter it). Using characteristic functions, we note that we can rewrite $\Phi_{\sqrt{M}S_m}(t)$ as

$$\begin{aligned}
\Phi_{\sqrt{M}S_M}(t) &= \mathbb{E}\left(e^{it\sqrt{M}S_M}\right) \\
&= \mathbb{E}\left(e^{it\sqrt{M}\left(\frac{1}{M}\sum_m \mathcal{E}(X^{(h,m)}) - \mathbb{E}(\mathcal{E}(X^{(h)}))\right)}\right) \\
&= \mathbb{E}\left(e^{it\sqrt{M}\left(\frac{1}{M}\sum_m \mathcal{E}(X^{(h,m)})\right)}\right) \\
&= \mathbb{E}\left(e^{it\mathcal{E}(X^{(h,1)})/\sqrt{M}}\cdots e^{it\mathcal{E}(X^{(h,M)})/\sqrt{M}}\right) \\
&= \left(\mathbb{E}\left(e^{it\mathcal{E}(X^{(h)})/\sqrt{M}}\right)\right)^M = \Phi_{t\mathcal{E}(X^{(h)})}(1/\sqrt{M})^M.
\end{aligned}$$

Now we apply Taylor expansion to get

$$\begin{aligned}\left(\Phi_{t\mathcal{E}(X^{(h)})}(1/\sqrt{M})\right)^M &= \left(1 + \frac{1}{\sqrt{M}}\Phi'_{t\mathcal{E}(X^{(h)})}(0) + \frac{1}{2M}\Phi''_{t\mathcal{E}(X^{(h)})}(0) + o\left(\frac{1}{M}\right)\right)^M \\ &= \left(1 - \frac{1}{2M}\mathrm{Var}(t\mathcal{E}(X^{(h)})) + o\left(\frac{1}{M}\right)\right)^M \xrightarrow{M\to\infty} e^{-\frac{1}{2}t^2\mathrm{Var}(\mathcal{E}(X^{(h)}))},\end{aligned}$$

where we in the second step used the relation

$$\mathbb{E}(X^k) = i^{-k}\Phi_X^{(k)}(0),$$

and in the last step the limit $(1 + a/x)^x \to e^a$.

ii) Assume that $Mh^2 = \mathrm{Cst} \neq 0$, and prove a central limit theorem but with a non-centered Gaussian random variable at the limit. For this, we assume that the weak error can be expanded at order 1 w.r.t. h.

Solution: This time we can neither neglect the statistical nor the discretization error. Instead, note that the discretization error is deterministic and denote it by $D_h \sim_c h$ (since it is the weak error). Hence, we write

$$\mathrm{Error}_{h,M} = S_M + D_h \sim_c S_M + Ah,$$

for some constant A. Next, we note that if we try to repeat previous calculations, we get

$$\mathbb{E}\left(e^{it\sqrt{M}(S_M+Ah)}\right) = e^{it\sqrt{M}Ah}\mathbb{E}\left(e^{it\sqrt{M}S_M}\right) \xrightarrow{M\to\infty} e^{it\mathrm{A}\cdot\sqrt{\mathrm{Cst}}-\frac{1}{2}t^2\mathrm{Var}(\mathcal{E}(X^{(h)}))},$$

where we have the constant because of the relation $\sqrt{M}h = \sqrt{\mathrm{Cst}}$.