Exercise 6.2

(Multi-level method with various strong convergence order)

Assume that the strong convergence of the Euler scheme is of order 1 w.r.t. h, and the weak convergence order is still 1 w.r.t h.

i) By a similar analysis to that of Theorem 6.3.1, determine the optimal allocation of computational effort within the different levels (as a function of number of simulations).

<u>Solution:</u> Repeating the calculations on page 200 in the course literature, we get similar results but with a few other details. The multi-level Monte-Carlo estimator is defined as

$$\overline{f(X_T)}^{h_0,\dots,h_L}_{M_0,\dots,M_L} = \frac{1}{M_0}\sum_{m=1}^{M_0} f(X_T^{h,0,m}) + \sum_{l=1}^{L}\frac{1}{M_l}\sum_{m=1}^{M_l}\big(f(X_T^{(h_l,l,m)}) - f(X_T^{(h_{l-1},l-1,m)})\big).$$

To derive an expression for the variance of the estimator, we first note that we can write the variance of a Monte-Carlo estimator as

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left(\frac{1}{M}\sum_{m=1}^{M} f(X_T^{(m)})\right) &= \mathbb{E}\left(\left(\frac{1}{M}\sum_{m=1}^{M} f(X_T^{(m)}) - \mathbb{E}(f(X_T))\right)^2\right) \\
&= \frac{1}{M^2}\mathbb{E}\left(\left(\sum_{m=1}^{M}\big(f(X_T^{(m)}) - \mathbb{E}(f(X_T))\big)\right)^2\right) \\
&= \frac{1}{M^2}\sum_{m=1}^{M}\mathbb{E}\Big(\big(f(X_T^{(m)}) - \mathbb{E}(f(X_T))\big)^2\Big) = \frac{1}{M}\mathbb{V}\mathrm{ar}\big(f(X_T)\big),
\end{aligned}$$

since the mixed terms become zero due to independence between simulations. The variance is thus written as

$$\mathbb{V}\mathrm{ar}\big(\overline{f(X_T)}^{h_0,\dots,h_L}_{M_0,\dots,M_L}\big) = \frac{\mathbb{V}\mathrm{ar}\big(f(X_T^{(h_0)})\big)}{M_0} + \sum_{l=1}^{L}\frac{\mathbb{V}\mathrm{ar}\big(f(X_T^{(h_l)}) - f(X_T^{(h_{l-1})})\big)}{M_l},$$

by using the independence of the simulations between the levels (to split the $\mathbb{V}\mathrm{ar}$ between the levels) and within the levels (to rewrite each estimator as above). One numerator within the sum can moreover be estimated as

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\big(f(X_T^{(h_l)}) - f(X_T^{(h_{l-1})})\big) &\le \mathbb{E}\big(\big|f(X_T^{(h_l)}) - f(X_T^{(h_{l-1})})\big|^2\big) \\
&\le 2\mathbb{E}\big(\big|f(X_T^{(h_l)}) - f(X_T)\big|^2\big) + 2\mathbb{E}\big(\big|f(X_T^{(h_{l-1})}) - f(X_T)\big|^2\big) \\
&= \mathcal{O}(h_l^2) + \mathcal{O}(h_{l-1}^2) = \mathcal{O}(h_l^2),
\end{aligned}$$

since in this case we have the strong convergence of order 1 instead of 1/2. We thus get an upper bound on the squared quadratic error

$$\begin{aligned}
\mathbb{E}\left(\mathrm{Error}^2_{h,M}\right) &:= \mathbb{E}\left(\big|\overline{f(X_T)}^{h_0,\dots,h_L}_{M_0,\dots,M_L} - \mathbb{E}\big(f(X_T)\big)\big|^2\right) \\
&= \mathbb{V}\mathrm{ar}\big(\overline{f(X_T)}^{h_0,\dots,h_L}_{M_0,\dots,M_L} - \mathbb{E}\big(f(X_T)\big)\big) + \left(\mathbb{E}\big(\overline{f(X_T)}^{h_0,\dots,h_L}_{M_0,\dots,M_L}\big) - \mathbb{E}\big(f(X_T)\big)\right)^2 \\
&\le_c h_L^2 + \frac{\mathbb{V}\mathrm{ar}\big(f(X_T^{(h_0)})\big)}{M_0} + \sum_{l=1}^{L}\frac{h_l^2}{M_l},
\end{aligned}$$

where in the last step the weak convergence order of 1 was applied. At the same time, the cost of the algorithm is (denoting the cost on each level by $\mathcal{C}_{\text{cost}}^{(l)} \sim_c M_l(h_l^{-1} + h_{l-1}^{-1}) \sim_c M_l h_l^{-1}$)

$$\mathcal{C}_{\text{cost}} = \sum_{l=0}^{L} \mathcal{C}_{\text{cost}}^{(l)} \sim_c \sum_{l=0}^{L} \frac{M_l}{h_l}.$$

Comparing these two, we find the relationship (lowering the error and cost at the same rate) $M_l = h_l^{3/2}$, so that the rate is given as

$$M_l = M_0 \cdot 2^{-3l/2}.$$

This result can be seen by applying Lagrange multiplier method on the problem

$$\begin{aligned} &\text{minimize} \quad \mathcal{C}_{\text{cost}} =: f(M_1, ..., M_L), \\ &\text{subject to} \quad \sum_{l=1}^{L} \frac{h_l^2}{M_l} = \varepsilon^2, \end{aligned}$$

i.e. construct the Lagrangian $\mathcal{L}(M_1, ...M_L, \lambda) = f + \lambda g$, where g is given by the constraint, and then analyzing the partial derivatives,

$$\frac{\partial \mathcal{L}}{\partial M_l} = 0 \implies \frac{1}{h_l} - \lambda \frac{h_l^2}{M_l^2} = 0 \implies M_l = \lambda h_l^{3/2},$$

and since $M_l = M_0 \cdot h_l^{cl}$, we conclude that we set $\lambda = M_0$ (more rigorously, this can be seen by looking at $\partial\mathcal{L}/\partial\lambda = 0$ and noting that $\lambda \sim \varepsilon^{-2}$, and then noting that $M_0 \sim \varepsilon^{-2}$ in next part, so we set $\lambda = M_0$). (follows if we set $\lambda = M_0$ in the end).

ii) What is the global complexity $\mathcal{C}_{\text{cost}}$ as a function of the tolerance error ε?

Solution: By the above expression

$$\mathcal{C}_{\text{cost}} \sim_c \sum_{l=0}^{L} \frac{M_l}{h_l} = M_0 \sum_{l=0}^{L} h_l^{1/2} = M_0 \sum_{l=0}^{L} \left(\frac{1}{\sqrt{2}}\right)^l = M_0 \frac{1 - (1/\sqrt{2})^{L+1}}{1 - 1/\sqrt{2}} < M_0(2 + \sqrt{2}) =: M_0 C,$$

where we applied the formula for geometric sum, bounded it by passing $L \to +\infty$ and rewrote the limit. Moreover, we wish to have

$$\sqrt{\mathbb{E}(\text{Error}_{h,M}^2)} \leq_c 2^{-L} + \sqrt{\frac{\tilde{C} + C}{M_0}} < \varepsilon,$$

where $\tilde{C} = \mathbb{V}\text{ar}(f(X_T^{(h_0)}))$. For this to be satisfied, we note that $2^{-L} = \varepsilon$ gives us $L = |\log(\varepsilon)|/\log(2)$, and for the square root expression we note that it suffices to choose $M_0 \sim_c \varepsilon^{-2}$. One difference in this result in comparison with the book (when we do it with lower order of strong convergence), is that we can skip the L within the square root expression, and hence get better asymptotic choice for M_0.

iii) More generally, assume that the Euler scheme converges strongly at order $\alpha \in (0, 1]$, and weakly of order $\beta \in (0, 1]$ (observe that $\alpha \leq \beta$). Derive the complexity/accuracy analysis associated with a multi-level method. What are the configurations of (α, β) for which (after optimizing the effort within levels)

$$\mathcal{C}_{\text{cost}} \sim_c \varepsilon^{-2},$$

i.e. we retrieve the standard Monte-Carlo convergence rate?

<u>Solution:</u> Similar calculations, but with convergence orders α and β give us

$$\mathbb{E}\left(\text{Error}_{h,M}^2\right) \leq_c h_L^{2\beta} + \frac{\mathbb{V}\text{ar}\big(f(X_T^{(h_0)})\big)}{M_0} + \sum_{l=1}^{L} \frac{h_l^{2\alpha}}{M_l},$$

and for the cost we still have

$$\mathcal{C}_{\text{cost}} \sim_c \sum_{l=0}^{L} \frac{M_l}{h_l}.$$

Applying the Lagrange multiplier method once more, we find the more general result

$$M_l = M_0 h_l^{\alpha+1/2} = M_0 \cdot 2^{-(\alpha+1/2)l}.$$

The sum in the quadratic error thus becomes

$$\frac{1}{M_0}\sum_{l=1}^{L} h_l^{2\alpha-(\alpha+1/2)} = \frac{1}{M_0}\sum_{l=1}^{L} h_l^{\alpha-1/2}.$$

and the cost can be written as

$$\mathcal{C}_{\text{cost}} \sim_c M_0 \sum_{l=1}^{L} 2^{-l(\alpha-1/2)}. \tag{1}$$

In the case $\alpha = 1/2$ we get

$$h_L^{2\beta} = 2^{-2\beta L} = \varepsilon^2 \implies L = \frac{|\log(\varepsilon)|}{\beta\log(2)},$$
$$\frac{1}{M_0}\sum_{l=1}^{L} h_l^{\alpha-1/2} = \frac{L}{M_0} = \varepsilon^2 \implies M_0 = \varepsilon^{-2}L \sim_c \varepsilon^{-2}|\log(\varepsilon)|.$$

The sum in (1) is bounded by L in this case so we get the cost $\mathcal{C}_{\text{cost}} \sim_c \varepsilon^{-2}|\log(\varepsilon)|^2$ (the square comes from the L).

If $\alpha < 1/2$, we have the same relation on L, and the sum becomes (after rewriting the geometric sum)

$$\frac{1}{M_0}\sum_{l=1}^{L} h_l^{\alpha-1/2} = \frac{1}{M_0}\sum_{l=1}^{L} 2^{l(1/2-\alpha)} = \frac{2^{(1/2-\alpha)(L+1)}}{M_0} \sim_c \frac{2^{L(1/2-\alpha)}}{M_0}.$$

Using the relation on L we can rewrite the numerator as

$$2^{L(1/2-\alpha)} = \left(2^L\right)^{1/2-\alpha} = \exp\left(\frac{|\log(\varepsilon)|}{\beta\log(2)}\log(2)\right)^{1/2-\alpha} = \varepsilon^{\left(\frac{1-2\alpha}{\beta}\right)}.$$

Set the sum equal to ε^2 and we find that

$$M_0 = \varepsilon^{-2}\varepsilon^{\left(\frac{1-2\alpha}{\beta}\right)},$$

which gives a cost of $\mathcal{C}_{\text{cost}} \sim_c \varepsilon^{-3}\varepsilon^{\left(\frac{1-2\alpha}{\beta}\right)}$ (the extra ε comes from the sum in (1)).

At last, for $\alpha > 1/2$ we get (let $a \in (0, 1/2]$ here for convenient notation)

$$\frac{1}{M_0} \sum_{l=1}^{L} h_l^{\alpha - 1/2} = \frac{1}{M_0} \sum_{l=1}^{L} 2^{-al} = \frac{1}{M_0} \left(\frac{1 - 2^{-a(L+1)}}{1 - 2^{-a}} - 1 \right) \leq_c \frac{1}{M_0}.$$

Hence we get $M_0 \sim_c \varepsilon^{-2}$ and moreover $\mathcal{C}_{\text{cost}} \sim_c \varepsilon^{-2}$. We conclude that the desired result follows in the case of $\alpha \in (1/2, 1]$ and $\beta \in [\alpha, 1]$.