Recall:

(IVP) $\begin{cases} \dot{y}(t) = f(y(t)) & \text{(DE)} \\ y(0) = y_0 & \text{(IC)} \end{cases}$ y_0, f given, $t \in [0,1]$

y ??

(BVP) $\begin{cases} -u''(x) = g(x) & 0 < x < 1 & \text{(DE)} \\ u(0) = 0, \; u(1) = 0 & & \text{(BC)} \end{cases}$

(PDE) $\begin{cases} u_t(x,t) - u_{xx}(x,t) = h(x) & 0 < x < 1, \; 0 < t < 3 & \text{(DE)} \\ u(0,t) = 0, \; u(1,t) = 0 & 0 < t < 3 & \text{(BC)} \\ u(x,0) = \sin(x) & 0 < x < 1 & \text{(IC)} \end{cases}$

Elliptic, parabolic, hyperbolic

Laplace, heat, wave

Chapter II: Mathematical tools

Goal: Introduce / recall some definitions, tools, (abstract) spaces. These will be used later on.

1) Vector spaces:

Def: A set V of vectors or functions is called a vector space (VS) or linear space if

$\forall\, u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$, one has

(i) Associativity: $u+(v+w)=(u+v)+w=u+v+w$

(ii) Commutativity: $u+v=v+u$

(iii) Identity element: $\exists\, 0 \in V$ s.t. $u+0=0+u=u$

(iv) Inverse element: $\forall u \in V$, $\exists\, (-u) \in V$ s.t.
$u+(-u)=0$

(v) $\alpha(\beta v)=(\alpha\beta)v=\alpha\beta v$

(vi) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \cdot v = v \quad \forall v \in V$

(vii) $\alpha(u+v)=\alpha u+\alpha v$

(viii) $(\alpha+\beta)u=\alpha u+\beta u$

Ex: $V=\mathbb{R}^2=\{(x,y) : x,y\in\mathbb{R}\}$ is a vector space / linear space, with

$(x_1,y_1)+(x_2,y_2)=(x_1+x_2, y_1+y_2)$

$\alpha(x,y)=(\alpha x, \alpha y)$

Ex: $V = \mathbb{R}, \mathbb{R}^3, \mathbb{R}^n$ are other example

Ex: V = space of polynomials defined on $[0,1]$ of degree ≤ 1.

$$V = \{ p(x) : p(x) \text{ polynomial with } x \in [0,1], \deg(p(x)) \leq 1 \} =$$

$$= \{ p(x) : p(x) = a_0 + a_1 x \text{ for } a_0, a_1 \in \mathbb{R}, x \in [0,1] \}$$

With the operations

$$p(x) + q(x) = (a_0 + a_1 x) + (b_0 + b_1 x) =$$

$$= (a_0 + b_0) + (a_1 + b_1) x$$

$$\alpha\, p(x) = \alpha (a_0 + a_1 x) = (\alpha a_0) + (\alpha a_1) x$$

Verify (i)–(viii) above $\Rightarrow$ V is vector space.

Ex: Similarly, one verifies that

$\mathcal{P}^{(k)}([a,b])$ = set of polyn. defined on $[a,b]$ of degree $\leq k$

is a vector space. $\mathcal{P}^{(k)}(\mathbb{R})$ also V.S.

Def: Let V vector space. A subset $U \subset V$ is called a subspace of V if

$\alpha u + \beta v \in U \quad \forall u, v \in U, \forall \alpha, \beta \in \mathbb{R}$.

Ex: Let $V = \mathbb{R}^3$ and $U = \{(x,y,0) : x,y \in \mathbb{R}\}$

Is U a subspace of V?

(i) Subset? $U \subset V$ clear since $(x,y,0) \in \mathbb{R}^3$

(ii) $\alpha u + \beta v \in U$? Clear

Ex: Let $V = \mathcal{P}^{(5)}(\mathbb{R})$ and $U = \mathcal{P}^{(1)}(\mathbb{R})$.

Is U a subspace of V?

(i) Subset? polyn. in U have degree ≤ 1 and $1 \leq 5$

(ii) $\underbrace{\alpha p}_{\substack{\text{polyn} \\ \deg \leq 1}} + \underbrace{\beta q}_{\substack{\text{polyn.} \\ \deg \leq 1}} \in U$? The sum will also be a polyn. of degree ≤ 1.

Def: Let V be a vector space and $v_1, v_2, \dots, v_n \in V$. A linear combination of these elements is given by

$a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ for some $a_1, a_2, \dots, a_n \in \mathbb{R}$.

Def: Let V be a vector space and $v_1, v_2, \dots, v_n \in V$. The space of all linear combination of these elements is denoted by

$$\text{span}(v_1, v_2, \dots, v_n) = \{ a_1 v_1 + a_2 v_2 + \dots + a_n v_n : a_1, a_2, \dots, a_n \in \mathbb{R} \}$$

Ex: Consider polyn. $1, x, x^2$.

What is $\text{span}(1, x, x^2) = ?$

$$\text{span}(1, x, x^2) \underset{\uparrow}{=} \{ a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 : a_0, a_1, a_2 \in \mathbb{R} \}$$

Def span

$= \mathcal{P}^{(2)}(\mathbb{R})$

Def $\mathcal{P}^{(2)}$: set of polyn. of degree ≤ 2.

How can we write the polynomial $4x-2 \in \mathcal{P}^{(2)}(\mathbb{R})$ in terms of $1, x, x^2$?

$4x-2 = (-2)\cdot 1 + 4(x) + 0\cdot(x^2)$

"coordinates"

2) Basis of a vector space:

Let V be a VS/linear space.

Def: • A set $\{v_1, v_2, \dots, v_n\}$ in V is called linearly independant if the equation $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$ has only the trivial solution $a_1 = a_2 = \dots = a_n = 0$.

Else, the set is called linearly dependant

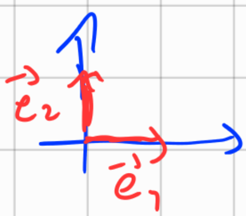
• A set $\{v_1, v_2, \dots, v_n\}$ in V is called a basis of V if $\text{span}(v_1, v_2, \dots, v_n) = V$

and the set is linearly independant.

- The dimension of V is denoted and defined as $\dim(V) = n$ = number of element in the basis.

Ex: • $V = \mathbb{R}^2$

• Basis $\{(0,1); (1,0)\}$



• $\dim(V) = \dim(\mathbb{R}^2) = 2$

Ex: $\dim \mathcal{P}^{(2)} = 2+1 = 3$ since $\text{span}(1, x, x^2) = \mathcal{P}^{(2)}$.

3) Inner product/scalar product:

Rem: $\mathbb{R}^2$ $\quad u^T v = u \cdot v = (u,v) = (u_1, u_2)^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2$

Let V be a linear space/vector space

Def: An inner product (scalar product) on V

is a map $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}$ s.t.

(i) $(u,v) = (v,u)$ $\qquad$ (symmetry)

(ii) $(u+\alpha v, w) = (u,w) + \alpha(v,w)$ (linearity)

(iii) $(v,v) \geq 0$ (positivity)

(iv) $(v,v) = 0 \Leftrightarrow v = 0$

$\forall \alpha \in \mathbb{R}, \forall u,v,w \in V$

Def: • A linear space V with an inner product is called an inner product space.

Notation $(V, (\cdot,\cdot))$ or $(V, (\cdot,\cdot)_V)$

• In such space, one has a norm defined by $\|u\|_V = \sqrt{(u,u)_V}$

Ex: $V = \mathbb{R}^2$, $u^T v$ as above

Norm $\|u\| = \sqrt{u_1^2 + u_2^2}$ for $u = (u_1, u_2) \in \mathbb{R}^2$

Def: Let $(V, (\cdot,\cdot))$ be an inner product space. 2 elements are orthogonal in V if $(u,v) = 0$, notation $u \perp v$.

4) Spaces of square integrable functions:

Def: Let $\Omega \subset \mathbb{R}^n$, we define the space of square integrable functions by

$$L^2(\Omega) = \left\{ f: \Omega \to \mathbb{R} : \int_\Omega |f(x)|^2 dx < \infty \right\}$$

(measurable)

with inner product given by

$$(f,g)_{L^2(\Omega)} = (f,g)_{L^2} = \int_\Omega f(x) \cdot g(x)\, dx$$

and the L^2-norm

$$\|f\|_{L^2(\Omega)} = \|f\|_{L^2} = \sqrt{(f,f)_{L^2}} = \sqrt{\int_\Omega |f(x)|^2 dx}$$

Rem: Think $\Omega = [0,1]$: $(f,g)_{L^2} = \int_0^1 f(x) g(x)\, dx$

vectors $\mathbb{R}^n$ $\quad u^T v = \sum_{i=1}^n u_i v_i$

$$\|f\|_{L^2} = \sqrt{\int_0^1 |f(x)|^2 dx}$$

$$\|u\| = \sqrt{\sum u_i^2}$$

Ex: Consider 2 polynomials $p(x) = x$ and $q(x) = x^2$ defined on $[-1,1]$. Are they orthogonal?

$p(x) \perp q(x) \Leftrightarrow (p,q)_{L^2(-1,1)} = 0 \Leftrightarrow$

$$\int_{-1}^{1} p(x)q(x)dx = \int_{-1}^{1} x \cdot x^2 dx = \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0$$

and hence $p(x) \perp q(x)$!

The above can be generalised

<u>Def</u>: • Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^n$. We define the spaces

$$\underline{L^p(\Omega)} = \{ f : \Omega \to \mathbb{R} \ : \ \|f\|_{L^p} < \infty \}, \text{ where}$$

the <u>L^p-norm</u> is defined by $\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}$

• Let $p = \infty$. We define the space

$$\underline{L^\infty(\Omega)} = \{ f : \Omega \to \mathbb{R} \ : \ \|f\|_\infty < \infty \}, \text{ where}$$

the <u>L^∞-norm</u>/maximum norm is defined by

"Think sup/max"

$$\|f\|_{L^\infty} = \operatorname*{ess.sup}_{x \in \Omega} |f(x)|$$

[essential supremum]

↳ wiki if interested

For us if Ω compact and f continuous, then $\|f\|_{L^\infty} = \max_{x \in \Omega} |f(x)|$