Recall:

- Wave eq.

$$(W)\begin{cases} u_{tt}(x,t)-u_{xx}(x,t)=f(x,t) & 0<x<1 \\ u(0,t)=u(1,t)=0 & 0<t\leq T \\ u(x,0)=u_0(x),\ u_t(x,0)=v_0(x) \end{cases}$$

$u_t = v$

$\Leftrightarrow$ $$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

- Conservation of energy ($f \equiv 0$)

$$\frac{1}{2}\|u_t(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|u(\cdot,t)\|_{L^2}^2 \equiv \text{Const} \quad \forall t \geq 0$$

- Variational formulation:

(VF) Find $u(\cdot,t)\in H_0^1(0,1)$ for $0<t\leq T$, s.t.

$$(u_{tt}(\cdot,t),v)_{L^2} + (u_x(\cdot,t),v')_{L^2} = (f(\cdot,t),v)_{L^2} \ \forall v\in H_0^1$$

$$u(x,0)=u_0(x),\ u_t(x,0)=v_0(x)$$

(ii) To get a FE problem, consider the space $V_h^0 = \text{span}(\varphi_1, \varphi_2, \dots, \varphi_m)$, where $\varphi_j$ are hat functions. We get

(FE) Find $u(\cdot, t) \in V_h^0$ for $0 < t \leq T$ s.t.

$$(u_{tt}(\cdot,t), \chi)_{L^2} + (u_x(\cdot,t), \chi')_{L^2} = (f(\cdot,t), \chi)_{L^2} \quad \forall \chi \in V_h^0$$

$$u(x,0) = \Pi_h u_0(x)$$

$$u_t(x,t) = \Pi_h v_0(x)$$

interpolant of $u_0, v_0$ $(\in V_h^0)$

(iii) From (FE), we will find a system of linear differential equation by writing

$$u(x,t) = \sum_{j=1}^{m} \boxed{\xi_j(t)}\, \varphi_j(x) \quad \text{and take } \chi = \varphi_i \text{ for } i = 1, \dots, m.$$

time dependant

Into (FE) gives:

$$\sum_{j=1}^{m} \ddot{\xi}_j(t) \underbrace{(\varphi_j, \varphi_i)}_{m_{ij}} + \sum_{j=1}^{m} \xi_j(t) \underbrace{(\varphi_j', \varphi_i')}_{S_{ij}} = \overbrace{(f(\cdot,t), \varphi_i)}^{F_i(t)}$$

$$\forall i = 1, 2, \dots, m.$$

The above is a system of linear DE:

$M\ddot{\xi}(t) + \mathcal{S}\,\xi(t) = F(t)$, where

$M \rightarrow$ mass matrix $(m \times m)$

$\mathcal{S} \rightarrow$ stiffness matrix $(m \times m)$

$F \rightarrow$ load vector $(m \times 1)$

$\xi(t) \rightarrow$ unknown $(m \times 1)$

With initial value $\xi(0) = \xi_0$, $\dot{\xi}(0) = \eta_0$,

where
$$\xi_0 = \begin{pmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_m) \end{pmatrix} \text{ and } \eta_0 = \begin{pmatrix} v_0(x_1) \\ \vdots \\ v_0(x_m) \end{pmatrix}.$$

Next, we rewrite the above DE as a system of first order DE:

$$\begin{cases} M\dot{\xi}(t) = M\eta(t) \\ M\dot{\eta}(t) + \mathcal{S}\,\xi(t) = F(t) \end{cases}$$

(iv) Apply a numerical method to the above system: C-N with stepsize $k$.

$$\begin{cases} M\left(\dfrac{\zeta^{(n+1)}-\zeta^{(n)}}{k}\right) = M\left(\dfrac{\eta^{(n+1)}+\eta^{(n)}}{2}\right) \\ M\left(\dfrac{\eta^{(n+1)}-\eta^{(n)}}{k}\right) + S'\left(\dfrac{\zeta^{(n+1)}+\zeta^{(n)}}{2}\right) = \dfrac{F(t_{n+1})+F(t_n)}{2} \end{cases}$$

Or as block-matrix:

$$\begin{pmatrix} M & -\frac{k}{2}M \\ \frac{k}{2}S' & M \end{pmatrix}\begin{pmatrix} \zeta^{(n+1)} \\ \eta^{(n+1)} \end{pmatrix} = \begin{pmatrix} M & \frac{k}{2}M \\ -\frac{k}{2}S' & M \end{pmatrix}\begin{pmatrix} \zeta^{(n)} \\ \eta^{(n)} \end{pmatrix} + $$

$$+ \begin{pmatrix} 0 \\ \frac{k}{2}\left(F(t_{n+1})+F(t_n)\right) \end{pmatrix}.$$

This provides approximations

$$\zeta^{(n)} \approx \zeta(t_n)$$
$$\eta^{(n)} \approx \dot{\zeta}(t_n) = \eta(t_n)$$

for $t_n = nk$.

Let us show

<u>Th</u>: If $f \equiv 0$ in $(W)$, then C-N preserves a discrete energy!

Rem: Backward/forward Euler schemes do not!!

Proof:

• C-N reads:

$$M\left(\frac{\zeta^{(n+1)}-\zeta^{(n)}}{k}\right) - M\left(\frac{\eta^{(n+1)}+\eta^{(n)}}{2}\right)=0 \qquad \Big|\cdot k\left(\zeta^{(n+1)}+\zeta^{(n)}\right)^T S' M^{-1}$$

$$M\left(\frac{\eta^{(n+1)}-\eta^{(n)}}{k}\right)+S'\left(\frac{\zeta^{(n+1)}+\zeta^{(n)}}{2}\right)=0 \qquad \Big|\cdot k\left(\eta^{(n+1)}+\eta^{(n)}\right)^T$$

+ add

$$\left(\zeta^{(n+1)}+\zeta^{(n)}\right)^T S'\left(\zeta^{(n+1)}-\zeta^{(n)}\right) - \frac{k}{2}\left(\zeta^{(n+1)}+\zeta^{(n)}\right)^T S'\left(\eta^{(n+1)}+\eta^{(n)}\right)$$

$$+\left(\eta^{(n+1)}+\eta^{(n)}\right)^T M\left(\eta^{(n+1)}-\eta^{(n)}\right) + \frac{k}{2}\left(\eta^{(n+1)}+\eta^{(n)}\right)^T S'\left(\zeta^{(n+1)}+\zeta^{(n)}\right)=$$

$$=0.$$

$= 0$ since $S'$ symmetric

Since $S'$ and $M$ are symmetric, the above reduces to:

$$\left(\zeta^{(n+1)}\right)^T S'\left(\zeta^{(n+1)}\right)+\left(\eta^{(n+1)}\right)^T M\left(\eta^{(n+1)}\right) =$$

$$= \left(\zeta^{(n)}\right)^T S'\left(\zeta^{(n)}\right)+\left(\eta^{(n)}\right)^T M\left(\eta^{(n)}\right)$$

$$= \|\, \underbrace{\|u_x^{(n)}\|_{L^2}^2}_{\text{potential}} + \underbrace{\|u_t^{(n)}\|_{L^2}^2}_{\text{kinetic}}$$

$$\|u_x^{(n)}\|_{L^2}^2 = \left( \sum_{j=1}^{m} \xi_j^{(n)} \varphi_j'(x), \sum_{i=1}^{m} \xi_i^{(n)} \varphi_i'(x) \right)_{L^2} =$$

$$= \sum_{j,i} \xi_j^{(n)} \xi_i^{(n)} \underbrace{(\varphi_j', \varphi_i')_{L^2}}_{S_{ij}} = (\xi^{(n)})^T S' \xi^{(n)}$$

## Chapter X: On our way to FEM in 2d

Goal: Extend integration by parts in 2d and higher.

Extend pw linear interpolation, FEM.

### 1) Green's formula:

To get a VF in 1d, we started with integration by parts $\rightarrow$ we extend this: Green's formula.

Th: (Green's formula)

Let $\Omega \subset \mathbb{R}^2$ open and bounded with $\partial\Omega$ pw smooth.

For $u \in C^{(2)}(\Omega)$ and $v \in C^{(1)}(\Omega)$, one get

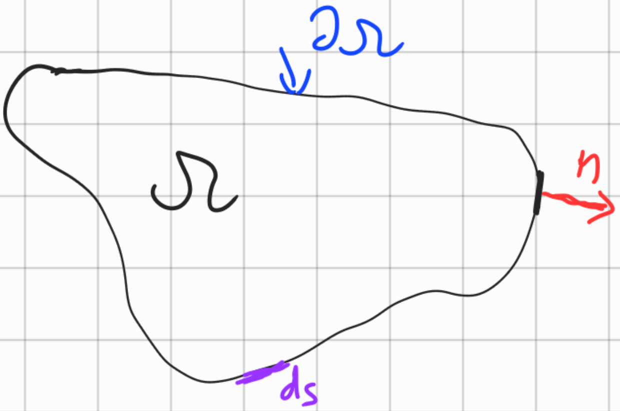
$$\iint_\Omega \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) v \, dx\,dy = \int_{\partial\Omega} \left( \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot n \right) v \, ds$$

$$- \iint_\Omega \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) dx\,dy,$$

where $n = n(x,y)$ is the outward unit normal vector at the boundary at the point $(x,y) \in \partial\Omega$;

$ds$ is a curve element on $\partial\Omega$,

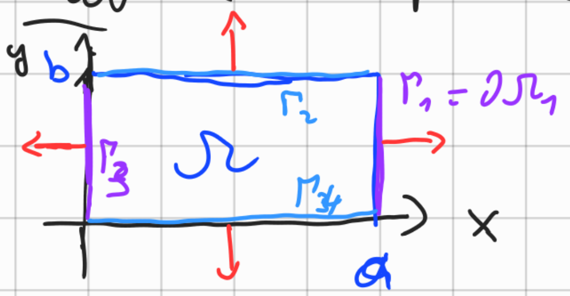
$\cdot$ stand for scalar product, dot product, inner prod.




In short:

$$\int_\Omega (\Delta u) v \, d\underset{\substack{\uparrow \\ x = (x_1, x_2) \in \mathbb{R}^2}}{x} = \int_{\partial\Omega} (\nabla u \cdot n) v \, ds - \int_\Omega \nabla u \cdot \nabla v \, dx$$

Proof: ("simple case")




- Let us start with

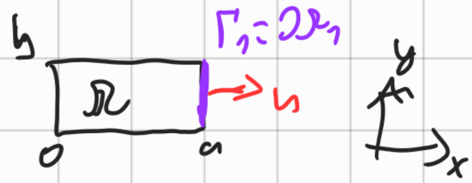
$$\iint_\Omega \frac{\partial^2 u}{\partial x^2}(x,y)\, v(x,y)\, dx\, dy \underset{\text{Def } \Omega}{=} \int_0^b \left( \int_0^a \frac{\partial^2 u}{\partial x^2}(x,y)\, v(x,y)\, dx \right) dy =$$

$$\underset{\text{int. by parts in 1d}}{=} \int_0^b \left( \frac{\partial u}{\partial x}(a,y)\, v(a,y) - \frac{\partial u}{\partial x}(0,y)\, v(0,y) \right) dy$$

$$- \int_0^b \left( \int_0^a \frac{\partial u}{\partial x}(x,y)\, \frac{\partial v}{\partial x}(x,y)\, dx \right) dy =$$

$$\underset{\text{Def } \Omega}{=} \dots - \iint_\Omega \frac{\partial u}{\partial x}(x,y)\, \frac{\partial v}{\partial x}(x,y)\, dx\, dy \qquad (*)$$

- Next, observe that, on $\Gamma_1 = \partial\Omega_1$




one has $n(x,y) = (1,0)$, x is constant $\to$ $\cancel{dx}$.

We compare with term, see above, (*)

$$\int_0^b \frac{\partial u}{\partial x}(a,y)\, v(a,y)\, dy = \int_0^b \underset{\substack{\| \\ (1,0)}}{n(x,y)} \cdot \left(\frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)\right) v(x,y)\, dy =$$

on $\Gamma_1$ : $x = a$ "$\forall x \in \Gamma_1$"

$$= \int_{\Gamma_1} (n \cdot \nabla u)\, v\, ds$$

- Similarly, on $\Gamma_3$, one has $n(x,y) = (-1,0)$ and (see (*)):

$$\int_0^b \left(-\frac{\partial u}{\partial x}(0,y)\, v(0,y)\right) dy = \ldots = \int_{\Gamma_3} (n \cdot \nabla u)\, v\, ds$$

$\Rightarrow$ Thus, we can rewrite (*) as follows:

$$\iint_\Omega \frac{\partial^2 u}{\partial x^2}\, v\, dx\, dy = \int_{\Gamma_1 \cup \Gamma_3} (n \cdot \nabla u)\, v\, ds - \iint_\Omega \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}\, dx\, dy \qquad (1)$$

- Similarly:

$$\iint_\Omega \frac{\partial^2 u}{\partial y^2}\, v\, dx\, dy = \int_{\Gamma_2 \cup \Gamma_4} (n \cdot \nabla u)\, v\, ds - \iint_\Omega \frac{\partial u}{\partial y} \frac{\partial v}{\partial y}\, dx\, dy \qquad (2)$$

Finally, taking the sum of (1) and (2), we get

$$\int_\Omega (\Delta u)\, v\, dx = \int_{\partial\Omega} (n \cdot \nabla u)\, v\, ds - \int_\Omega (\nabla u) \cdot (\nabla v)\, dx$$