Recall:

- (BVP) $\begin{cases} -(a(x)u'(x))' = f(x), & 0 < x < 1 \quad \text{DE} \\ u(0)=0,\ u(1)=0 & \text{hom. Dirichlet BC} \end{cases}$

(VF) Find $u \in H_0^1$ s.t. $\int_0^1 a(x)u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v \in H_0^1$

(FE) Find $u_h \in V_h^0$ s.t. $\int_0^1 a(x)\,u_h'(x)\,v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v \in V_h^0$

- A priori error estimate for above problem:

$\|u - u_h\|_E \leq C \cdot \|h u''\|_a$,

where $\|f\|_E = \sqrt{(f', f')_a}$ and $(f,g)_a = \int_0^1 a(x)\, f(x) g(x)\,dx$

$h(x)$ mesh function.

- Interpolation error $\|f - \pi_h f\|_{L^p} \leq C \cdot \|h f'\|_{L^p}$ (Chap. V)

$\pi_h f$: cont. pw linear interpolant of f

$\pi_h f(x_j) = f(x_j)$

We now show an a posteriori error estimate for the above (BVP) in energy norm.

<u>Th</u>: Under some technical assumptions on

u (sol. (VF)) and u_h (FE-sol. given by c.G(1)),

set $e := u - u_h$ for error and define

→ rest of (BVP) when inserting u_h

the residual $R(u_h(x)) = f(x) + (a(x) u_h'(x))'$,

then we have the following a posteriori error estimates

$$\|e\|_E = \|u - u_h\|_E \leq C \Bigg(\overbrace{\int_0^1 \frac{1}{a(x)} h^2(x) R^2(u_h(x)) \, dx}^{\text{can compute!}} \Bigg)^{1/2}.$$

⟶ 0 as $h \to 0$

<u>Proof</u>:

- First, we observe that $\underset{\uparrow \atop H_0^1}{e = u} - \underset{\uparrow \atop V_h^0 \subset H_0^1}{u_h} \in H_0^1$ and

hence can test with $e \in H_0^1$ in (VF). This gives us

$$\int_0^1 a(x) u'(x) e'(x) \, dx = \int_0^1 f(x) e(x) \, dx \qquad (*)$$

• Next, compute the error:

$$\|e\|_E^2 \underset{\text{Def } \|\cdot\|_E}{=} \int_0^1 a(x)\, e'(x) \cdot e'(x)\, dx \underset{e=u-u_h}{=} \int_0^1 a(x) \left(u'(x) - u_h'(x)\right) e'(x)\, dx =$$

$$= \int_0^1 a(x) u'(x) e'(x)\, dx - \int_0^1 a(x) u_h'(x) e'(x)\, dx =$$

$$\overset{(*)}{=} \int_0^1 f(x) \underbrace{e(x)\, dx}_{-\Pi_h e + \Pi_h e} - \int_0^1 a(x) u_h'(x) \underbrace{e'(x)\, dx}_{-(\Pi_h e)' + (\Pi_h e)'}$$

• We now insert and remove the interpolant

$\Pi_h e$ and its derivative $(\Pi_h e)'$ to get:

$$\|e\|_E^2 = \int_0^1 f(x) \left(e(x) - \underline{\Pi_h e(x)}\right) dx + \underbrace{\int_0^1 f(x)\, \underline{\Pi_h e(x)}\, dx}$$

$$- \int_0^1 a(x) u_h'(x) \left(e'(x) - \underline{(\Pi_h e)'}(x)\right) dx - \underbrace{\int_0^1 a(x) u_h'(x) (\underline{\Pi_h e})'(x)\, dx}$$

Since u_h is sol. to (FE) and $\Pi_h e \in V_h^0$, the sum of the red terms is zero! Thus, the error reads:

$$\|e\|_E^2 = \int_0^1 f(x) \left(e(x) - \Pi_h e(x)\right) dx - \int_0^1 a(x) u_h'(x) \left(e' - (\Pi_h e)'\right) dx$$

We leave the first as it is and work with second : $\int_0^1 \rightsquigarrow \sum_{j=1}^{M+1} \int_{x_{j-1}}^{x_j}$ and do int. by parts

We get:

$$\|e\|_E^2 = \cdots - \sum_{j=1}^{M+1} \int_{x_{j-1}}^{x_j} a(x) u_h'(x) \left(e(x) - \Pi_h e(x)\right)' dx$$

by parts

$$= \cdots - \sum_{j=1}^{M+1} \left\{ a(x) u_h'(x) \underbrace{\left(e(x) - \Pi_h e(x)\right)}_{=0 \text{ because of interpolation!}} \Big|_{x_{j-1}}^{x_j} - \int_{x_{j-1}}^{x_j} \left(a(x) u_h'(x)\right)' \left(e(x) - \Pi_h e(x)\right) dx \right. =$$

$$= \int_0^1 f(x) \left(e(x) - \Pi_h e(x)\right) dx + \int_0^1 \left(a(x) u_h'(x)\right)' (e - \Pi_h e) dx$$

$$= \int_0^1 \Big(\underbrace{f(x) + \left(a(x) u_h'(x)\right)'}_{R(u_h(x)) \text{ residual}} \Big) \left(e(x) - \Pi_h e(x)\right) dx$$

$$= \int_0^1 R(u_h(x)) \left(e(x) - \Pi_h e(x) \right) dx$$

- In order to get back the energy norm we add and remove some stuffs.

$$\|e\|_E^2 = \int_0^1 \frac{1}{\sqrt{a(x)}}\, h(x)\, R(u_h(x))\, \sqrt{a(x)}\, \frac{\left(e(x)-\pi_h e(x)\right)}{h(x)}\, dx$$

C-S in L^2

$$\le \left(\int_0^1 \frac{1}{a(x)}\, h^2(x)\, R(u_h(x))^2\, dx \right)^{1/2} \cdot$$

$$\cdot \underbrace{\left(\int_0^1 a(x) \left(\frac{e(x)-\pi_h e(x)}{h(x)} \right)^2 dx \right)^{1/2}}_{\left\| \frac{e-\pi_h e}{h} \right\|_a} \le C \cdot \|e'\|_a \le C \cdot \|e\|_E$$

by Def $\|.\|_a$

interpolation error Chap IV, Rem. 5.3, book (last eq. recall p. 1)

Def $\|.\|_E$

This gives us

$$\|e\|_E^{\cancel{2}} \le C \cdot \left(\int_0^1 \frac{1}{a(x)}\, h^2(x)\, R^2(u_h(x))\, dx \right)^{1/2} \cdot \cancel{\|e\|_E}$$

$$\hookrightarrow \|e\|_E \le C \cdot \left(\int_0^1 \frac{1}{a(x)}\, h^2(x)\, R^2(u_h(x))\, dx \right)^{1/2}$$

$\blacksquare$

4) Adaptivity:

Recall the a posteriori bound

$$\| u - u_h \|_E \leq C \cdot \underbrace{\left(\int_0^1 \frac{1}{a(x)} h(x)^2 R^2(u_h(x)) dx \right)^{1/2}}_{(*)}$$

Someones comes and want the error of c(h/1) to be smaller than a given tolerance

$$\| u - u_h \|_E \overset{!}{\leq} TOL,$$

TOL given (think $10^{-2}, 10^{-3}$)

Idea:

(i) Start with some preliminary mesh $h(x)$ $\rightsquigarrow$ compute FE sol. $u_h(x)$.

(ii) Compute $(*)$.

If $(*) \leq TOL$:-) Ok $\to$ error $\leq TOL$

If $(*) > TOL$ $\rightsquigarrow$

refine locally the mesh

(for instance where $\frac{1}{\sigma(x)} R(u_h(x))^2$ is

large) $\rightsquigarrow$ get new mesh $h(x)$

$\rightsquigarrow$ compute new FE sol. $u_h(x)$

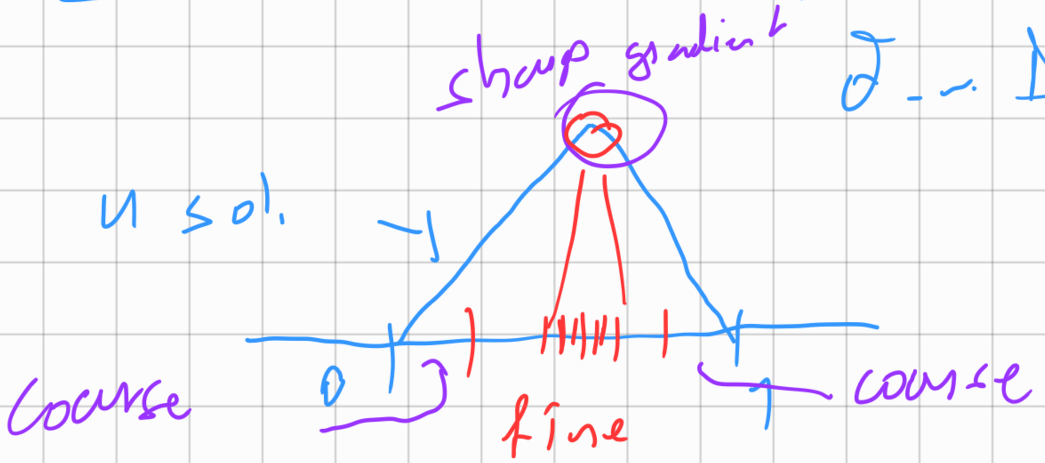
(iii) If $(*) > TOL \rightsquigarrow$ (ii)

If not :-)

:-(Costly (takes computational time)

:-) Good for problems with shocks or irregular sol.

Ex: $-u''(x) = \delta(x)$, where δ ... Dirac



Book Larsson/Bengzon p. 59

5) FEM for convection-diffusion-absorption BVP:

We derive FEM for various BVP and highlight the needed modifications.

We start with the BVP

$$(\mathrm{BVP})\quad \begin{cases} -u''(x) + 4u(x) = 0 & \text{for } 0<x<1 \text{ (DE)} \\ u(0) = \alpha,\ u(1) = \beta, & \text{BC} \end{cases}$$

$\alpha, \beta \neq 0$ given.

The above BC are called <u>non-homogeneous</u> <u>Dirichlet</u> BC.

(i) In order to find the weak formulation/variational formulation, we consider the <u>trial space</u> (where solution lives)

$$V = \{ v : (0,1) \to \mathbb{R} \,;\, v \in H^1(0,1) \text{ and } v(0) = \alpha,\ v(1) = \beta \}$$

the <u>test space</u> (where test functions live)

$$V^0 = \{ v : [0,1] \to \mathbb{R} \,;\, v \in H_0^1(0,1) \}.$$

Next, multiply (BVP) with test function $v \in V^0$, integrate by part to get:

$$\underbrace{-\int_0^1 u''(x)\, v(x)\, dx}_{\underbrace{-u'(x)v(x)\Big|_0^1}_{=0 \text{ since } v \in V^0 = H_0^1.} + \int_0^1 u'(x)\, v'(x)\, dx} + 4\int_0^1 u(x)\, v(x)\, dx = 0 \quad \forall v \in V^0$$

This gives us the variational formulation

Find $u \in \underline{V}$ s.t. $\int_0^1 u'(x) v'(x)\, dx + 4\int_0^1 u(x) v(x)\, dx = 0 \quad \forall v \in \underline{V^0}$

(ii) Next, we derive a FE problem.

Consider partition of $[0,1]$,

$T_h: x_0 = 0 < x_1 < x_2 < \dots < x_m < x_{m+1} = 1$

with a uniform mesh $h = \frac{1}{m+1}$ (simple notation)

Consider the following spaces:

$V_h = \{ v: [0,1] \to \mathbb{R} : v$ cont. pw linear on T_h and $v(0) = \alpha, v(1) = \beta \}$

$V_h^0 = \{ v: [0,1] \to \mathbb{R} : v$ cont. pw linear on T_h and $v(0) = 0, v(1) = 0 \}$

Observe that

$$V_h^0 = \mathrm{span}(\varphi_1, \varphi_2, \dots, \varphi_m) \rightsquigarrow \dim(V_h^0) = m$$

$$V_h = \mathrm{span}(\varphi_0, \varphi_1, \dots, \varphi_m, \varphi_{m+1}) \rightsquigarrow \dim(V_h) = m+2.$$

This gives us the FE problem:

(FE) Find $u_h \in V_h$ s.t. $\int_0^1 u_h'(x) \chi'(x)\, dx + 4 \int_0^1 u_h(x) \chi'(x)\, dx$

$\forall \chi \in V_h^0$

(iii) Finally, we need to find a linear system.

To do this, we first write

$$u_h(x) = \sum_{j=0}^{m+1} \xi_j \varphi_j(x)$$

unknowns; hat fct

But we must have $u_h(0) \overset{!}{=} \alpha$ (non-h. Dirichlet)

$u_h(1) \overset{!}{=} \beta$

$$u_h(0) = \alpha \iff \sum_{j=0}^{m+1} \zeta_j \varphi_j(0) = \alpha \iff$$

$$\zeta_0 \underbrace{\varphi_0(0)}_{=1} + \zeta_1 \underbrace{\varphi_1(0)}_{=0} + \dots + \zeta_{m+1} \underbrace{\varphi_{m+1}(0)}_{=0} \overset{!}{=} \alpha$$

by def φ_j hat

$$\iff \zeta_0 \overset{!}{=} \alpha \,!$$

Similarly, $\zeta_{m+1} \overset{!}{=} \beta$!

$$\Rightarrow u_h(x) = \alpha \varphi_0(x) + \sum_{j=1}^{m} \zeta_j \varphi_j(x) + \beta \varphi_{m+1}(x).$$

We insert the above into (FE) and take test function $\chi(x) = \varphi_i(x)$ for $i = 1, \dots, m$ to obtain:

$$\left(\alpha \varphi_0' + \sum_{j=1}^{m} \zeta_j \varphi_j' + \beta \varphi_{m+1}', \varphi_i'\right)_{L^2} +$$

$$+ 4\left(\alpha \varphi_0 + \sum_{j=1}^{m} \zeta_j \varphi_j + \beta \varphi_{m+1}, \varphi_i\right)_{L^2} = 0 \quad \forall i = 1, 2, \dots m.$$