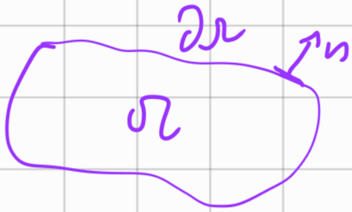
Recall:

- Energy wave: $\frac{1}{2} \|u_t(\cdot,t)\|^2_{L^2(0,1)} + \frac{1}{2} \|u_x(\cdot,t)\|^2_{L^2(0,1)}$

- $\Omega \subset \mathbb{R}^2$, u, v nice $\rightsquigarrow$ Green's formula:

$$\int_\Omega \Delta u \, v \, dx = \int_{\partial\Omega} (n \cdot \nabla u) v \, ds - \int_\Omega \nabla u \cdot \nabla v \, dx$$



2) Variational formulation in 2d:

Let $\Omega \subset \mathbb{R}^2$ nice domain and $f: \Omega \to \mathbb{R}$ nice

(ex: $f \in L^2(\Omega)$). Consider Poisson's equation in 2d:

$$(P) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$(\Leftrightarrow u(x_1, x_2) = 0$ for $(x_1, x_2) \in \partial\Omega)$

boundary

As for BVP in 1d, we multiply the DE with a test function v (nice) and vanishing on $\partial\Omega$.

Integrate over Ω and use Green's formula:

$$\int_\Omega f v \, dx = -\int_\Omega \Delta u \, v \, dx \underset{\text{Green's formula}}{=} -\int_{\partial\Omega} (n \cdot \nabla u) v \, ds + \int_\Omega \nabla u \cdot \nabla v \, dx$$

$= 0$ since $v = 0$ on $\partial\Omega$

$$\Leftrightarrow \underbrace{\int_\Omega f v \, dx}_{\ell(v)} = \underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}_{a(u,v)} \Leftrightarrow \ell(v) = a(u,v)$$

We get the VF for (P):

(VF) Find $u \in H_0^1(\Omega)$ s.t. $a(u,v) = \ell(v) \quad \forall v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$$

Ex: • Above integrals make sense:

$$\int_\Omega f v \, dx \underset{C-S}{\leq} \|f\|_{L^2(\Omega)} \cdot \|v\|_{L^2(\Omega)} < \infty \text{ if f.ex } \begin{matrix} f \in L^2 \\ v \in L^2 \end{matrix}$$

- One can also use Lax-Milgram to show $\exists!$ of a weak solution (sol. of (VF)).
- To get equivalence between weak and strong sol. one needs

  sol. to (VF) ↲ ↳ sol. to (P)

  regularity assumptions on Ω and possibly on f.

3) Triangulation:

Motivation: 1d: $[a,b] \rightsquigarrow$ partition

2d:  $\rightsquigarrow$ triangulation

From now on, we assume that $\Omega \subset \mathbb{R}^2$ is bounded with polygonal boundary $\partial\Omega$ (or smooth boundary).

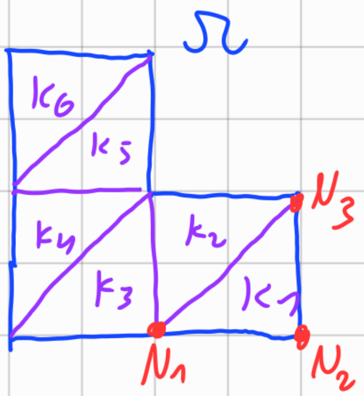
Def: • A <u>triangulation</u>, or <u>mesh</u>, of Ω is denoted by T_h and consist of a set $\{K\}$ of triangles K s.t.

$$\Omega = \bigcup_{K\in T_h} K \quad \text{and} \quad K_1 \cap K_2 = \begin{cases} \text{empty} \\ \text{edge} \\ \text{corner} \end{cases}$$

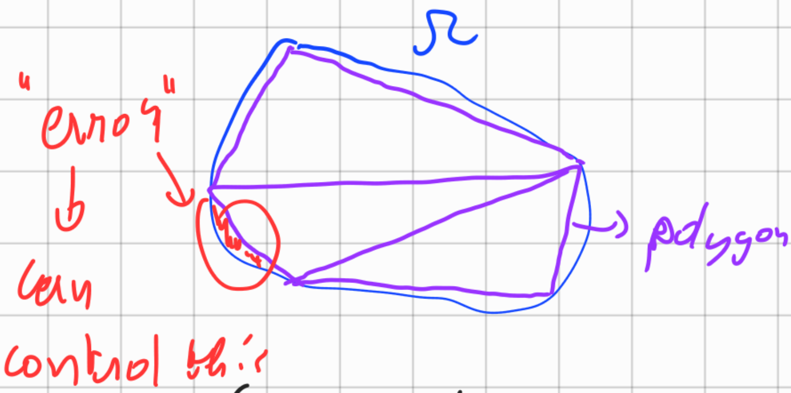
(no triangle corner is hanging, i.e. cannot be on an edge of another triangle)

- The corners of a triangle are called <u>nodes</u>
- The <u>local mesh size</u> of a triangle K is h_K is length of the longest edge of K.
- The <u>global mesh size</u> is denoted by h and given by $h = \max_{K\in T_h} h_K$.

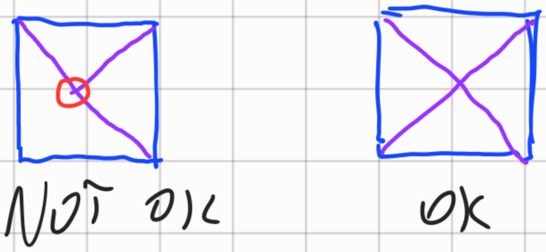
Ex:



L-shape Ω



Smooth domain



Rem: • Any polygon has a triangulation:

Fan triangulation, Euler.

• Mesh generators to automatically build a mesh of Ω

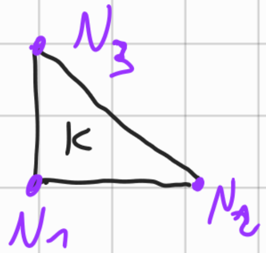
• 2D: Delaunay mesh generator $\rightsquigarrow$ matlab pde toolbox.

• 3D: Possible but more involved.

Rem: We will always assume that all triangles are regular.

4) The space of linear polynomials:

Consider a triangle K with nodes N_1, N_2, N_3:



Def: The space of linear polynomials on K is defined by

$$P_1(K) = \{ v : K \to \mathbb{R} : v(x_1, x_2) = c_0 + c_1 x_1 + c_2 x_2 \text{ for } (x_1, x_2) \in K \text{ and } c_0, c_1, c_2 \in \mathbb{R} \}$$

Rem: Any $v \in P_1(K)$ is uniquely determined by its nodal values $\alpha_j = v(N_j) = v(\overbrace{x_1^{(j)}, x_2^{(j)}}^{\text{coordinates of } N_j})$ for $j = 1, 2, 3$:

$$\alpha_1 \overset{!}{=} c_0 + c_1 x_1^{(1)} + c_2 x_2^{(1)} \quad \rightsquigarrow \text{1 eq. for 3 unknowns}$$

$$\alpha_2 = c_0 + c_1 x_1^{(2)} + c_2 x_2^{(2)}$$

$$\alpha_3 = c_0 + c_1 x_1^{(3)} + c_2 x_2^{(3)}$$

↳ 3 eq. for 3 unknowns c_0, c_1, c_2.

$\rightsquigarrow$ $\exists!$ sol. c_0, c_1, c_2 (K non-degenerate)

Instead of the "classical" basis $\{1, x_1, x_2\}$ one prefers to consider nodal basis $\{\lambda_1, \lambda_2, \lambda_3\}$ defined by the conditions

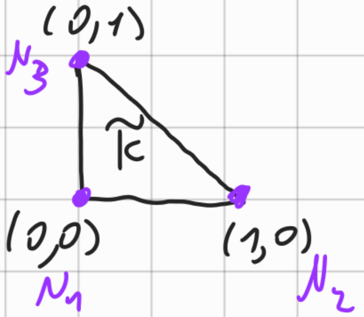
$$\lambda_j(N_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$(\text{for } i, j = 1, 2, 3)$$

Hence, any $v \in P_1(K)$ can be written as

$$v = \sum_{j=1}^{3} \alpha_j \lambda_j, \quad \text{where } \alpha_j = v(N_j).$$

For the reference triangle

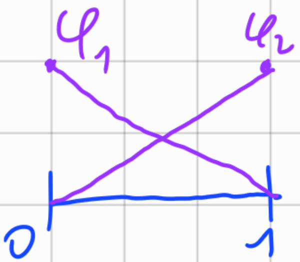


one has the following hat functions

$$\lambda_1(x_1, x_2) = 1 - x_1 - x_2$$

$$\lambda_2(x_1, x_2) = x_1 \qquad \text{for } (x_1, x_2) \in \tilde{K}$$

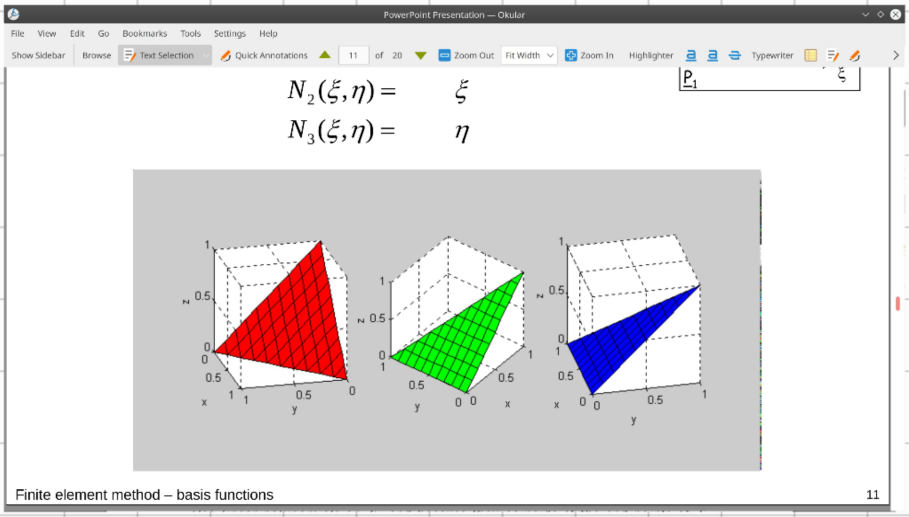
$$\lambda_3(x_1, x_2) = x_2$$



hat functions

1d ⇝ reference intervall

Ex: Plot of $\lambda_1, \lambda_2, \lambda_3$:



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5) The space of continuous piecewise linear functions:

Consider $\Omega \subset \mathbb{R}^2$ with polygonal boundary with triangulation/mesh $T_h = \{K\}$.

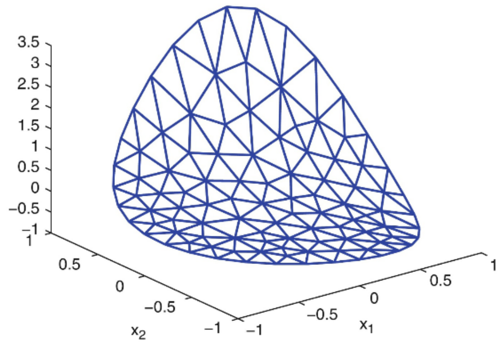
Def: The space of continuous piecewise linear functions is defined by

$$V_h = \{ v \in C^0(\Omega) : v|_K \in P_1(K) \ \forall K \in T_h \}$$

Ex:

3.2 Piecewise Polynomial Spaces 51

Fig. 3.3 A continuous piecewise linear function v



(c) Book by M. Larsson and Bengzon

Again a basis for the space V_h is given by the hat functions / pyramid functions $\{\varphi_j\}_{j=1}^{n_p}$, where n_p is number of nodes in the triangulation T_h of Ω.

φ_j are given by the conditions

$$\varphi_j(N_i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

for $i, j = 1, \dots, n_p$.

Ex:

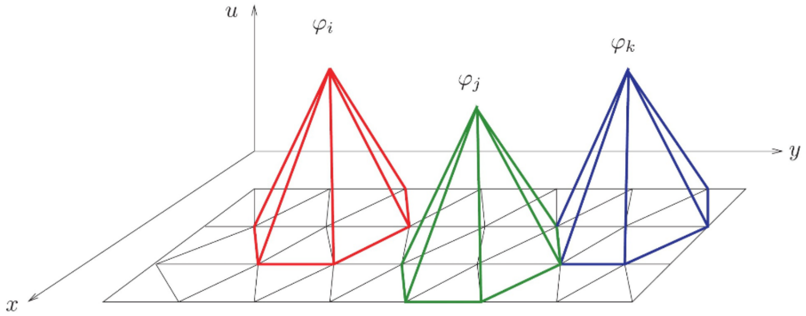


Figure 5.4. *Examples of hat functions in two dimensions.*

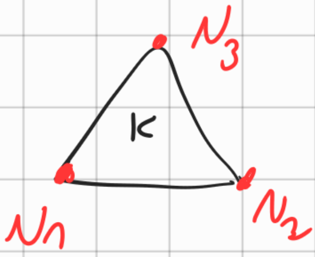
© Gander, Kwok.

Again, any $v \in V_h$ can be written as

$$v = \sum_{j=1}^{n_p} \alpha_j \varphi_j \quad , \text{ where } \alpha_j = v(N_j) \text{ for } j = 1, 2, \ldots, n_p .$$

6) Linear interpolation:

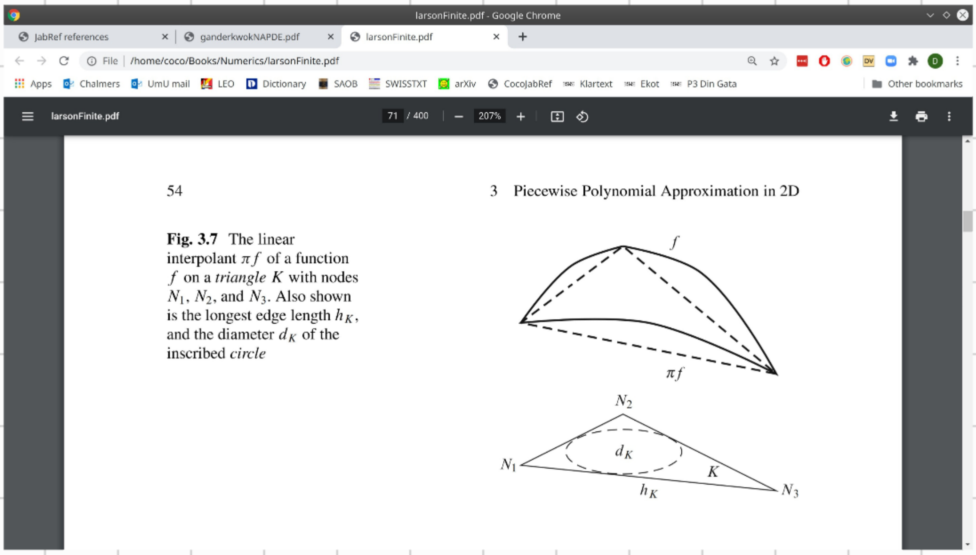
Consider a continuous function f defined on triangle K with nodes N_1, N_2, N_3:



Def: The linear interpolant of f, denoted by $\Pi_1 f \in \mathbb{P}_1(K)$, is defined by

$$\Pi_1 f = \sum_{j=1}^{3} f(N_j)\varphi_j$$

Rem: $\Pi_1 f \in P_1(K)$ is plane



We have the following estimates:

Thm: Let $f \in H^2(\Omega)$, then one has the error estimates for the linear interpolant $\Pi_1 f$:

$$\|\Pi_1 f - f\|_{L^2(K)} \le C_K \cdot h_K^2 \cdot \|f\|_{H^2(K)}$$

$$\|\nabla(\Pi_1 f - f)\|_{L^2(K)} \le \hat{C}_K \cdot h_K \cdot \|f\|_{H^2(K)},$$

where we recall that h_K is the local mesh size of K.

[1d: $\|\Pi_h f - f\|_{L^2(0,1)} \leq C \cdot h^2 \|f''\|_{L^2(0,1)}$]

(Proof uses Bramble-Hilbert lemma)

7) Continuous pw linear interpolant

Let $\Omega \subset \mathbb{R}^2$ domain polynomial boundary and $f: \Omega \to \mathbb{R}$ continuous. Denote by T_h a triangulation of Ω and recall that V_h is the space of cont pw linear functions on T_h.

Def. The continuous piecewise linear interpolant of f, denoted by $\Pi_h f$, is defined as

$$\Pi_h f = \sum_{j=1}^{n_p} f(N_j) \cdot \varphi_j$$

Rem: Matlab ⇝ pdesurf functions

We get the following result:

Th: Let $f \in H^2(\Omega)$ and T_h as above. Then, one has error estimates

$$\| \Pi_h f - f \|^2_{L^2(\Omega)} \leq C \sum_{K \in T_h} h_K^4 \, \|f\|^2_{H^2(\Omega)}$$

$$\| \nabla (\Pi_h f - f) \|^2_{L^2(\Omega)} \leq C \cdot \sum_{K \in T_h} h_K^2 \, \|f\|^2_{H^2(\Omega)}$$