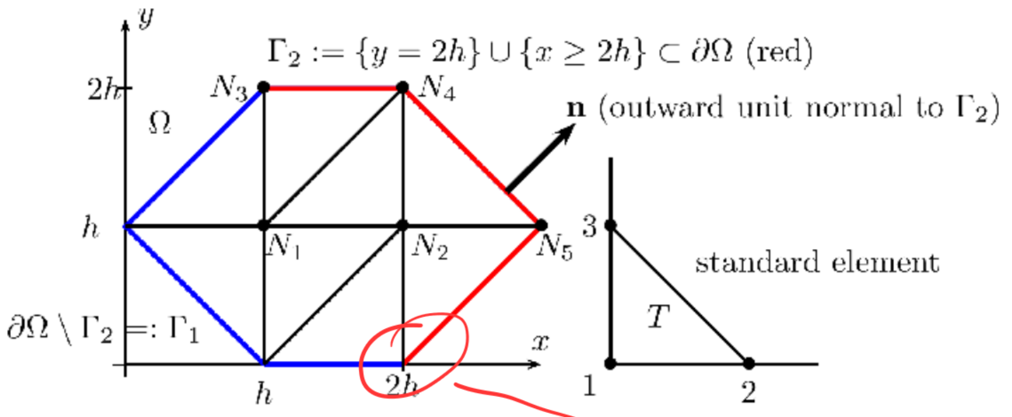
Exam 2019-03-20:

1. Let $\pi_h f$ be the linear interpolant of f in (a, b). Prove the $L_p(a, b)$ interpolation error estimates:

$$||\pi_h f - f||_{L_p(a,b)} \le (b-a)^2 ||f''||_{L_p(a,b)}, \qquad p = 1, 2.$$

2. Let Ω be the hexagonal domain with the uniform triangulation as in the figure below. Compute



the stiffness matrix and the load vector for the cG(1) approximate solution for the problem:

(1) $$-\Delta u = 1, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \Gamma_1, \quad \partial u/\partial \mathbf{n} = 1, \quad \text{on } \Gamma_2$$

? Nodes?

a) Derive variational formulation:

Consider space $V = \{ v \in H^1(\Omega) \text{ s.t. } u|_{\Gamma_1} = 0 \}$

and test DE with $v \in V$:

$$\int_\Omega 1 \cdot v \, dx = -\int_\Omega \Delta u \, v \, dx \underset{\uparrow}{=}$$

Green's formula

$$= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \underbrace{(n \cdot \nabla u)}_{= \frac{\partial u}{\partial n}} v \, ds =$$

$$\underset{\uparrow}{=} \int_\Omega \nabla u \cdot \nabla v \, dx \underbrace{- \int_{\Gamma_1} \frac{\partial u}{\partial n} v \, ds}_{= 0 \text{ since } v \in V} - \int_{\Gamma_2} \underbrace{\frac{\partial u}{\partial n}}_{= 1 \text{ by def BC}} v \, ds$$

$\partial\Omega = \Gamma_1 \cup \Gamma_2$

a) We obtain VF:

(VF) Find $u \in V$ s.t. $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega v \, dx + \int_{\Gamma_2} v \, ds \quad \forall v \in V$

b) Find finite element problem:

Consider $V_h = \{ v \text{ cont. pw linear on } T_h, \ v|_{\Gamma_1} = 0 \} \subset V$,

where T_h is the triangulation given in the exercise. We get the FE problem:

(FE) Find $u_h \in V_h$ s.t. $\int_\Omega \nabla u \cdot \nabla \chi \, dx = \int_\Omega \chi \, dx + \int_{\Gamma_2} \chi \, ds$

$\forall \chi \in V_h$

c) To get a linear system, consider

5 → ?? interation

$u_h(x) = \sum_{j=1} \xi_j \varphi_j(x)$ and $\chi(x) = \varphi_i(x)$ for $i = 1, 2, \dots, 5$

φ_j ⤳ hat functions.

⤳ $S \cdot \xi = b$, where

$S = (S_{ij})_{i,j=1}^5$ and $b = (b_i)_{i=1}^5$

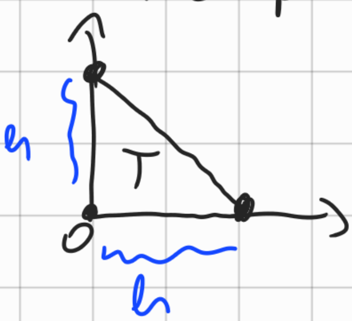
where

$$S_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx$$

$$b_i = \int_\Omega \varphi_i \, dx + \int_{\Gamma_2} \varphi_i \, ds$$

d) Finally, we compute these entries S_{ij} and b_i

On the physical domain Ω, one has



$\leadsto$ hat functions are given by

$$\varphi_1(x,y) = 1 - \frac{x}{h} - \frac{y}{h}$$

$$\varphi_2(x,y) = \frac{x}{h}$$

$$\varphi_3(x,y) = \frac{y}{h}$$

The element stiffness matrix (3×3) is given by

$$s = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}$$

(small s)

with $s_{11} = \int_T \nabla\varphi_1 \cdot \nabla\varphi_2 \, dx$ SEE END OF DOC

3. Derive a *posteriori* error estimate, in the energy norm, definied as $||v||_E^2 := ||v'||^2 + ||v||^2$, for the $cG(1)$ approximation of the boundary value problem

$$-u''(x) + 2xu'(x) + 2u(x) = f(x), \quad 0 < x < 1, \quad u'(0) = 0, \quad u(1) = 0.$$

4. Formulate dG(0) scheme for $\dot{u}(t) + a(t)u(t) = 0,\ u(0) = u_0,\ a(t) > 0$, and prove the stability

$$|U_N|^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \le |u_0|^2.$$

Malin → w5.pdf

5. Consider the convection problem

$$\beta \cdot \nabla u + \alpha\, u = f \quad x \in \Omega; \qquad u = g \text{ for } x \in \Gamma_- := \{x \in \partial\Omega : \beta(x) \cdot \mathbf{n}(x) < 0\}.$$

Assume that $\alpha - \frac{1}{2}\nabla \cdot \beta \geq c > 0$. Prove the stability estimate

$$c\|u\|^2 + \int_{\Gamma_+} \mathbf{n} \cdot \beta u^2 \, dx \leq \frac{1}{c}\|f\|^2 + \int_{\Gamma_-} |\mathbf{n} \cdot \beta| g^2 \, dx, \qquad \Gamma_+ := \partial\Omega \setminus \Gamma_-.$$

Hint: Show first $2(\beta \cdot \nabla u, u) = \int_{\Gamma_+} \mathbf{n} \cdot \beta u^2 \, dx - \int_{\Gamma_-} |\mathbf{n} \cdot \beta| u^2 \, dx - ((\nabla \cdot \beta)u, u)$.

a) How to start?

→ Hint ...

→ Test the eq. with a particular test function in (VF); typical choices $u(x,t)$, $u_x(x,t)$, $u_t(x,t)$, ...

b) Stability ⇒ error estimate?

Not directly. But can be used in error analysis.

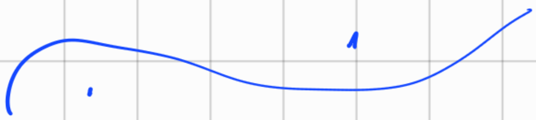
6. Consider the cG(1) approximation u_h for the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad \Omega \subset \mathbb{R}^d,\ d = 2, 3.$$

Let $e := u - u_h$ be the error of approximation and show the following gradient estimate in $L_2(\Omega)$:

$$\|\nabla e\| = \|\nabla(u - u_h)\| \leq C\|hD^2u\|.$$

$\rightsquigarrow$ see lecture (Chapter 11)



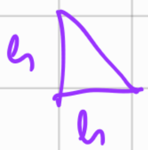
(Ex 2)

$$S_{11} = \int_T \nabla\varphi_1 \cdot \nabla\varphi_1 \, dxdy \underset{\varphi}{=} \int_T \left(-\frac{1}{h}, -\frac{1}{h}\right) \cdot \begin{pmatrix} -\frac{1}{h} \\ -\frac{1}{h} \end{pmatrix} dxdy =$$

$$\varphi_1(x,y) = 1 - \frac{x}{h} - \frac{y}{h}$$

$$= \int_T \frac{2}{h^2} \, dxdy = \frac{2}{h^2} \int_T dxdy = \frac{2}{h^2} |T| =$$

$$\underset{\varphi}{=} \frac{2}{h^2} \cdot \frac{h^2}{2} = 1$$



, $|T|$ = area of triangle

$$\ldots \qquad S = \frac{1}{2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

symmetric

Assembly of stiffness S: (symmetric)

$$S_{11} \underset{\uparrow}{=} 3 \cdot s_{11} + 1 \cdot s_{22} + 1 \cdot s_{33}$$

look at the nodes
N_1, N_1 on the exercise
and use the element
stiffness matrix s (small s)