Exam, 15 March 2021

TMA 372, MMG 800

1

a) No: discriminant $= 0 - 4\cdot 1\cdot 1^2 = -4 < 0 \Rightarrow$ PDE is elliptic 0,5 0,5

b) A posteriori error estimates 0,5, since it depends on u_h and computable 0,5 stuffs.

c) Chapter VI: $\|u-u_h\|_E \le C\cdot h\,\|u''\|_{L^2}$, with the energy norm $\|v\|_E = \sqrt{(v',v')_{L^2}}$ 1

d) One has $\varphi_i(x_j) = \delta_{ij}$ and thus $\varphi_i(x)\varphi_j(x) \neq 0$ only if i and j are not too far $\Rightarrow$ most of the entries in these matrices will be zero. 1

e) Let $u, v \in H^1$ and compute

$$|a(u,v)| \underset{\text{Def } H^1 \text{ norm}}{\overset{\Delta}{\le}} |(u,v)_{H^1}| + \left|\int_0^1 u'(x)v(x)\,dx\right| \overset{C-S}{\le} \|u\|_{H^1}\cdot\|v\|_{H^1} + \|u'\|_{L^2}\|v\|_{L^2} \le$$

$$\underset{\text{Def } H^1, L^2 \text{ norms}}{\le} 2\,\|u\|_{H^1}\,\|v\|_{H^1} \leadsto a \text{ is continuous}$$ 1

2

a) line 6 $\leadsto \begin{cases} y'(t) = y(t)^2 & 0,5 \\ y(0) = 1 & 0,5 \end{cases}$ $\quad 0 < t \le \cancel{10}\ 1$

b) line 6 $\leadsto$ explicit Euler scheme 1

c) $h = \frac{1}{10}$ 1

d) $t(1) = 0 + \frac{1}{10} = \frac{1}{10}$ 0,5 and $y(1) = 1 + \frac{1}{10}\cdot 1^2 = \frac{11}{10}$ 0,5 $\quad$ ok (t_0, y_0)

Since the question was not really well formulated

3

a) hom. Dirichlet BC at 0 $\leadsto$ consider $V = \{v:[0,1]\to\mathbb{R} : v\in H^1, v(0) = 0\}$ 0,5

(VF) Find $u \in V$ s.t. $(u',v')_{L^2} = (f,v)_{L^2} + 5\,v(1) \quad \forall v \in V$ 0,5

(using int. by part)

b) Consider $V_h = \{v:[0,1]\to\mathbb{R} : v$ cont. pw linear on T_h with $v(0)=0\} =$

$= \text{span}(\varphi_1, \varphi_2, \dots, \varphi_{m+1}) \subset V$, where 0,5

T_h unif. partition and φ_j hat functions

(FE) Find $u_h \in V_h$ s.t. $(u_h', \chi')_{L^2} = (f, \chi) + 5\,\chi(1) \quad \forall \chi \in V_h$. 0,5

c) Write $u_h(x) = \sum_{j=1}^{m+1} \xi_j \varphi_j(x)$ and take $v = \varphi_i$ into (FE) to get linear system

$$S\xi = b, \text{ where}$$

$b = (b_i)_{i=1}^{m+1}$ with $b_i = (f, \varphi_i) + \beta\, \varphi_i(1)$
$\uparrow$ zero except for $i = m+1$

0.5 0.5

④ Write $u = v + iw$ into Schrödingers:

$$iv_t - w_t = v_{xx} + i w_{xx} \iff \begin{cases} v_t = w_{xx} \\ w_t = -v_{xx} \end{cases}$$ $\cdot v$ and integrate $\int_0^1 \dots dx$, $\cdot w$ " " "

0.5

$$\iff \begin{cases} \int_0^1 v_t v\, dx = \int_0^1 w_{xx} v\, dx = -\int_0^1 w_x v_x\, dx \quad \text{(by part)} \\ \int_0^1 w_t w\, dx = -\int_0^1 v_{xx} w\, dx = \int_0^1 v_x w_x\, dx \end{cases}$$

$$\overset{\shortparallel}{\tfrac{1}{2}\tfrac{d}{dt}(w^2)}$$

0.5

0.5 0.5

Add both lines: $\frac{1}{2}\frac{d}{dt}\left(\|u(\cdot,t)\|^2_{L^2}\right) = 0$ hence $\|u(\cdot,t)\|_{L^2} = \|u_0\|_{L^2} = \text{const}$ $\forall t \geq 0$.

Or directly by computing $\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2} = \dots = 0$.

⑤

a) Test $f(x) = -\nabla\cdot(c(x)\nabla u(x)) + b(x)u(x)$ with $v \in H^1(\Omega)$ and use Green's formula f.ex. eq. 9.2.9 book

$$\int_\Omega f v\, dx = \int_\Omega c(x)\nabla u(x)\nabla v(x)\, dx - \underbrace{\int_{\partial\Omega} c(x)\frac{\partial u}{\partial n} v\, ds}_{=0 \text{ since BC}} + \int_\Omega b(x)u(x)v(x)\, dx \iff \ell(v) = a(u,v)$$

0.5

(VP) Find $u \in H^1$ s.t. $a(u,v) = \ell(v) \quad \forall v \in H^1$.

0.5

b) We want to use Lax-Milgram and need

$a(\cdot,\cdot)$ continuous: $|a(u,v)| \overset{C\text{-}S}{\leq} \|c\|_\infty \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \|b\|_\infty \|u\|_{L^2}\|v\|_{L^2}$ 1

$\rightsquigarrow$ ok if c, b continuous and bounded

$a(\cdot,\cdot)$ coercive: $a(u,u) = (c\nabla u, \nabla u)_{L^2} + (bu, u)_{L^2} \underset{\uparrow}{\geq}$ 1

if $c(x) \geq c_0 > 0$, $b(x) \geq b_0 > 0$

$\geq c_0\|\nabla u\|^2_{L^2} + b_0\|u\|^2_{L^2} \underset{\uparrow \text{Def } H^1 \text{ norm}}{\geq} d_0\|u\|^2_{H^1}$ where $d_0 = \min(c_0, b_0)$

$\hookrightarrow$ b, c cont., bounded, with $c(x), b(x) \geq d_0 > 0$.

1.5

(2)

b) ℓ continuous $\Rightarrow |\ell(v)| \overset{C\text{-}S}{\leq} \|f\|_{L^2} \|v\|_{L^2} \leq \text{Const} < \infty$ if $f \in L^2, v \in L^2$.

c) (FE) Find $u_h \in V_h$ (with $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$) s.t.

$$a(u_h, \chi) = \ell(\chi) \quad \forall \chi \in V_h, \text{ where}$$ (0.5)

$$V_h = \{ v \in C^0(\Omega) \text{ s.t. } v_{|K} \in P_1(K) \quad \forall K \in T_h \}.$$ (0.5)

d) matrix for node ②: $\int_\Omega 1 \cdot \varphi_2 \varphi_2 = \int \hat\varphi_1^2 + \int \hat\varphi_3^2 =: m$ (1)

Vector for node ①: $\int_\Omega 1 \cdot \varphi_1 = \int \hat\varphi_2 + \int \hat\varphi_3 =: b$, (1)

where reference hat functions: $\hat\varphi_1(x,y) = 1-x-y$

$\hat\varphi_2(x,y) = x$, $\hat\varphi_3(x,y) = y$

$$\int_\Delta \hat\varphi_1^2 = \int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy\, dx = \ldots = \frac{1}{12} = \int_\Delta \hat\varphi_3^2$$ (1)

$$\int_\Delta \hat\varphi_2 = \int_0^1 \int_0^{1-x} x \, dy\, dx = \ldots = \frac{1}{6} = \int_\Delta \hat\varphi_3$$ (1)

$\leadsto$ $m = \frac{2}{12} = \frac{1}{6}$ and $b = \frac{2}{6} = \frac{1}{3}$ //

⑥ (VF) Find $u \in H_0^1$ s.t. $\int_0^1 u'v'dx = \int_0^1 f v\, dx \quad \forall v \in H_0^1$

(FE) Find $u_h \in V_h^0$ s.t. $\int_0^1 u_h' v' dx = \int_0^1 f v\, dx \quad \forall v \in V_h^0$

$\xrightarrow{(GO)} \int_0^1 (u' - u_h') v' = 0 \quad \forall v \in V_h^0$

(AUX) Find $z \in H_0^1$ s.t. $\int_0^1 z' v' dx = \int_0^1 (u-u_h) v\, dx \quad \forall v \in H_0^1$ $(\Leftrightarrow -z'' = u - u_h)$

a) $\|u-u_h\|_{L^2}^2 = \int_0^1 (u-u_h)(u-u_h)\, dx \underset{(Aux)}{\overset{\in H_0^1}{=}} \int_0^1 z'(u'-u_h')\, dx \underset{\substack{(GO)\\ \pi_h z \in V_h^0}}{=} \int_0^1 (z-\pi_h z)'(u-u_h)'dx$ (1) (1)

b) $\|u-u_h\|_{L^2}^2 \overset{\text{above}}{=} \int_0^1 (z-\pi_h z)'(u-u_h)'dx \overset{C\text{-}S}{\leq} \|(z-\pi_h z)'\|_{L^2} \cdot \|(u-u_h)'\|_{L^2} \underset{\text{interp. error Chap. 5}}{\leq}$ (1)

$\leq Ch\|z''\|_{L^2} \|(u-u_h)'\|_{L^2} \underset{-z'' = (u-u_h)}{\leq} Ch\|u-u_h\|_{L^2} \|(u-u_h)'\|_{L^2}$ (1)

$\Rightarrow$ c) $\|u-u_h\|_{L^2} \leq C \cdot h \cdot \|(u-u_h)'\|_{L^2} \underset{\text{Def } H^1 \text{ norm}}{\leq} Ch\|u-u_h\|_{H^1} \underset{\text{Chapter 6}}{\leq} C \cdot h^2 \|u''\|_{L^2} \underset{-u''=f}{\leq} Ch^2 \|f\|_{L^2}$ (1) (1)