

**Examination, 26 August 2021**  
**MVE455**

**Read this before you start!**

*Aid: Anything but collaboration.*

*If something is not clear you can ask to talk to me over zoom.*

*In case of problems, I'll try to contact all of you via canvas.*

*Read all questions first and start to answer the ones you like most. Some parts of an exercise may be independent of the others.*

*Answers may be given in English, French, German or Swedish.*

*Write down all the details of your computations clearly so that each steps are easy to follow.*

*Do not randomly display equations and hope for someone to find the correct one. Justify your answers!*

*Write clearly what your solutions are and in the nicest possible form.*

*Don't forget that you can verify your solution in some cases.*

*Write your cid or first numbers of your personnummer.*

*Use a proper pen, order your answers, use an app like camscanner or equivalent, and check your final scan before uploading it. Check also the uploaded version.*

*The test has 4 pages and a total of 20 points.*

*Valid bonus points will be added to the total score if needed.*

*You will be informed when the exams are corrected.*

*"I assure that I did this exam on my own without getting help from any other person and that I formulated all the solutions myself."*

*Check the box ☐*

**Good luck!**

Some exercises were taken from, or inspired by, materials from H. Haddar, V. Heuveline, S.G. Rodriguez.

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1. Give the type (elliptic, parabolic, hyperbolic) of the following partial differential equations ( $u = u(x, y)$ ):

(a)  $\partial_x \partial_y u - \partial_x u = 0.$  (1p)

(b)  $\partial_x^2 u + \partial_x \partial_y u + y \partial_y^2 u + 4u = 0.$  (1p)

2. Let  $f \in L^2(0, 1)$  and consider the problem

$$\begin{aligned} -4u''(x) + (x - 1/2)u'(x) + 5u(x) &= f(x) \quad \text{for } x \in (0, 1) \\ u(0) &= 0, u(1) = 0. \end{aligned}$$

- (a) Find a functional space  $V$ , a bilinear form  $a: V \times V \rightarrow \mathbb{R}$  and a linear form  $\ell: V \rightarrow \mathbb{R}$  such that each solution  $u$  to the above problem satisfies the variational formulation

$$\text{Find } u \in V \text{ such that } a(u, v) = \ell(v) \text{ for all } v \in V. \quad (2p)$$

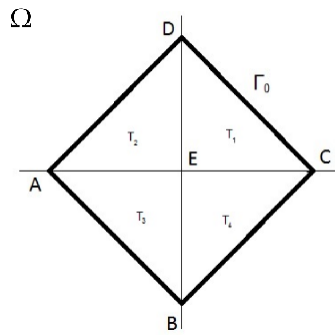


Figure 1: Courtesy from S.G. Rodriguez.

- (b) Show that the bilinear form  $a$  is coercive. (2p)
3. Consider the domain  $\Omega \subset \mathbb{R}^2$  in Figure 1. Set  $A = (-1, 0), B = (0, -1), C = (1, 0), D = (0, 1), E = (0, 0), \Gamma_0 = \overline{CD}, \Gamma_1 = \partial\Omega \setminus \Gamma_0$ . Let  $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$  and consider the variational problem

$$\text{Find } u \in V \text{ such that } a(u, v) = \ell(v) \text{ for all } v \in V,$$

where the bilinear and linear forms are defined by

$$a(u, v) = \int_{\Omega} \nabla u(x, y) \cdot \nabla v(x, y) \, dx \, dy \quad \text{and} \quad \ell(v) = \int_{\Omega} v(x, y) \, dx \, dy.$$

- (a) Define the FE space  $V_h$  for an approximation of the solution to this variational problem by cG(1) FE on the grid defined in the figure. (2p)
- (b) Provide a basis for the space  $V_h$  and write the FE solution  $u_h$  with help of the basis elements. (3p)
- Hint: A piecewise linear continuous function on  $\Omega$  is uniquely determined by its values at the nodes  $A, B, C, D, E$ . In this part of the question, you don't need to give an explicit formula for the basis elements, some precise explanatory text is enough.*
- (c) Give the final linear system  $M\zeta = L$  (3 eq. for 3 unknown) coming from the FE problem corresponding to the above variational problem. (3p)
- (d) Compute explicitly the first two diagonal terms in the matrix  $M$ . (3p)

*Hint: Here you need explicit formulas for the basis elements. If you are not able to derive explicit formulas for the basis functions on the nodes  $A, B, E$ , you can use the following formulas*

$$\varphi_A(x, y) = \begin{cases} 0 & \text{for } (x, y) \in T_1 \cup T_4 \\ -x & \text{for } (x, y) \in T_2 \cup T_3 \end{cases}$$

and

$$\varphi_B(x, y) = \begin{cases} 0 & \text{for } (x, y) \in T_1 \cup T_2 \\ -y & \text{for } (x, y) \in T_3 \cup T_4 \end{cases}$$

and

$$\varphi_E(x, y) = \begin{cases} -x - y + 1 & \text{for } (x, y) \in T_1 \\ x - y + 1 & \text{for } (x, y) \in T_2 \\ x + y + 1 & \text{for } (x, y) \in T_3 \\ -x + y + 1 & \text{for } (x, y) \in T_4. \end{cases}$$

Points will be deducted accordingly.

4. Let  $A < B$  and denote  $I = ]A, B[$ . Consider the variational problem: Find  $u \in H^1(I)$  such that

$$a(u, v) = (f, v) \quad \text{for all } v \in H^1(I)$$

and the corresponding cG(1) approximation: Find  $u_h \in V_h \subset H^1(I)$  such that

$$a(u_h, v) = (f, v) \quad \text{for all } v \in V_h.$$

One assumes that the bilinear form  $a$  is continuous, coercive and bounded on  $H^1(I)$  and that the given function  $f$  is nice enough.

Denote by  $\Pi_h: H^1(I) \rightarrow V_h$  the interpolation operator.

(a) Show that  $a(u_h - u, v) = 0$  for all  $v \in V_h$ . (1p)

(b) Next, show that  $\|\Pi_h u - u_h\|_{H^1(I)} \leq C \|\Pi_h u - u\|_{H^1(I)}$ . (2p)