Examination, 26 August 2021
TMA372 and MMG800

Read this before you start!
Aid: Anything but collaboration.
If something is not clear you can ask to talk to me over zoom.
In case of problems, I'll try to contact all of you via canvas.
Read all questions first and start to answer the ones you like most. Some parts of an exercise may be independent of the others.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for someone to find the correct one. Justify your answers!
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Write your cid or first numbers of your personnummer.
Use a proper pen, order your answers, use an app like camscanner or equivalent, and check your final scan before uploading it. Check also the uploaded version.
The test has 4 pages and a total of 30 points.
Valid bonus points will be added to the total score if needed.
You will be informed when the exams are corrected.

"I assure that I did this exam on my own without getting help from any other person and that I formulated all the solutions myself."
Check the box □
Good luck!

Some exercises were taken from, or inspired by, materials from *H. Haddar*, *V. Heuveline*, *S.G. Rodriguez*.

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1. Give the type (elliptic, parabolic, hyperbolic) of the following partial differential equations ($u = u(x, y)$):

   (a) $\partial_x\partial_y u - \partial_x u = 0.$ (1p)

   (b) $\partial_x^2 u + \partial_x\partial_y u + y\partial_y^2 u + 4u = 0.$ (1p)

   (c) $2(\partial_x + \partial_y)^2 u + \partial_y u = 0.$ (1p)

2. Let an integer N and two real numbers $a < b$. Consider a discretisation of the interval $[a, b]$, denoted $\{a = x_0 < x_1 < \ldots < x_N = b\}$, with

$$h_{\max} = \max_{k=0,\ldots,N-1} |x_{k+1} - x_k|.$$

   Show that for all continuous and piecewise continuously differentiable functions f with

$$f(x_0) = f(x_1) = \ldots = f(x_N) = 0,$$

one has
$$\|f\|_{L^2} \le h_{\max}\|f'\|_{L^2}. \qquad (2p)$$
Hint: One may use the fundamental theorem of calculus
$$f(x) = f(y) + \int_y^x 1 \cdot f'(z)\,dz$$
for any $y \le x \in [a,b]$.

3. Provide a numerical approximation of $\int_0^1 u(x)\,\mathrm{d}x$ knowing that $u(0) = 0$ and $u(1) = 1$. (2p)

4. Let $f \in L^2(0,1)$ and consider the problem
$$\begin{aligned} &-4u''(x) + (x - 1/2)u'(x) + 5u(x) = f(x) \quad \text{for } x \in (0,1) \\ &u(0) = 0, u(1) = 0. \end{aligned}$$
   (a) Find a functional space V, a bilinear form $a\colon V \times V \to \mathbb{R}$ and a linear form $\ell\colon V \to \mathbb{R}$ such that each solution u to the above problem satisfies the variational formulation
   $$\text{Find } u \in V \text{ such that } a(u,v) = \ell(v) \text{ for all } v \in V. \qquad (2p)$$
   (b) Show that the bilinear form a is coercive. (2p)
   (c) Can one apply Lax–Milgram to show existence and uniqueness of a solution to the above variational formulation? (2p)

5. Let u_0 and f be given functions. Assume that there are nice solutions, denoted $u_1(x,t)$ and $u_2(x,t)$, to the heat equation
$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0 < x < 1, 0 < t \le T \\ u(0,t) = 0, u(1,t) = 0 & 0 < t \le T \\ u(x,0) = u_0(x) & 0 < x < 1. \end{cases}$$
   Define $w = u_1 - u_2$.
   (a) Provide a linear partial differential equation for which w is a solution to. (1p)
   (b) Use a stability estimates from the lecture (Chapter 8) to show that $\|w(\cdot,t)\|_{L^2} = 0$ for all t. (1p)

6. Consider the domain $\Omega \subset \mathbb{R}^2$ in Figure 1. Set $A = (-1,0), B = (0,-1), C = (1,0), D = (0,1), E = (0,0), \Gamma_0 = \overline{CD}, \Gamma_1 = \partial\Omega \setminus \Gamma_0$. Let $V = \{v \in H^1(\Omega) : v = 0 \quad \text{on } \Gamma_0\}$ and consider the variational problem
$$\text{Find } u \in V \text{ such that } a(u,v) = \ell(v) \quad \text{for all } v \in V,$$
   where the the bilinear and linear forms are defined by
$$a(u,v) = \int_\Omega \nabla u(x,y) \cdot \nabla v(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{and} \quad \ell(v) = \int_\Omega v(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

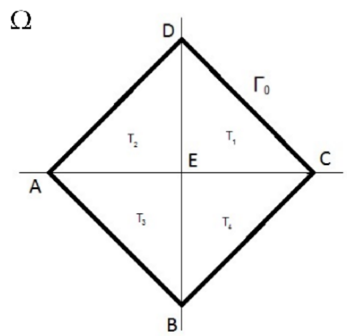

Figure 1: Courtesy from S.G. Rodriguez.

(a) Define the FE space V_h for an approximation of the solution to this variational problem by cG(1) FE on the grid defined in the figure. (2p)

(b) Provide a basis for the space V_h and write the FE solution u_h with help of the basis elements. (3p)

Hint: A piecewise linear continuous function on Ω is uniquely determined by its values at the nodes A, B, C, D, E. In this part of the question, you don't need to give an explicit formula for the basis elements, some precise explanatory text is enough.

(c) Give the final linear system $M\zeta = L$ (3 eq. for 3 unknown) coming from the FE problem corresponding to the above variational problem. (3p)

(d) Compute explicitely the first two diagonal terms in the matrix M. (3p)

Hint: Here you need explicit formulas for the basis elements. If you are not able to derive explicit formulas for the basis functions on the nodes A, B, E, you can use the following formulas

$$\varphi_A(x,y) = \begin{cases} 0 & \textit{for } (x,y) \in T_1 \cup T_4 \\ -x & \textit{for } (x,y) \in T_2 \cup T_3 \end{cases}$$

and

$$\varphi_B(x,y) = \begin{cases} 0 & \textit{for } (x,y) \in T_1 \cup T_2 \\ -y & \textit{for } (x,y) \in T_3 \cup T_4 \end{cases}$$

and

$$\varphi_E(x,y) = \begin{cases} -x-y+1 & \textit{for } (x,y) \in T_1 \\ x-y+1 & \textit{for } (x,y) \in T_2 \\ x+y+1 & \textit{for } (x,y) \in T_3 \\ -x+y+1 & \textit{for } (x,y) \in T_4. \end{cases}$$

Points will be deducted accordingly.

7. Let $A < B$ and denote $I =]A,B[$. Consider the variational problem: Find $u \in H^1(I)$ such that

$$a(u,v) = (f,v) \quad \text{for all } v \in H^1(I)$$

and the corresponding cG(1) approximation: Find $u_h \in V_h \subset H^1(I)$ such that

$$a(u_h, v) = (f, v) \quad \text{for all } \ v \in V_h.$$

One assumes that the bilinear form a is continuous, coercive and bounded on $H^1(I)$ and that the given function f is nice enough.

Denote by $\Pi_h \colon H^1(I) \to V_h$ the interpolation operator.

(a) Show that $a(u_h - u, v) = 0$ for all $v \in V_h$. (1p)

(b) Next, show that $||\Pi_h u - u_h||_{H^1(I)} \leq C||\Pi_h u - u||_{H^1(I)}$. (2p)

(c) Finally, use the triangle inequality to get the estimate $||u - u_h||_{H^1(I)} \leq C||\Pi_h u - u||_{H^1}$. (1p)