[TMA 372 / MMG 800]

Chapter 0: Introduction and motivation

see .pdf on canvas

Chapter I: Terminology

Goal: Introduce/recall notations/equations

Def: • A differential equation (DE) is an equation that relates an unknown function (or more) and its derivatives.

• An ordinary differential equation (ODE) is a DE, where the unknown fct depends on only one variable (say: $y(x)$, $u(t)$, ..

• A partial differential equation (PDE) is a DE, where the unknown fct depends on two (or more) independent variable (say $u(t,x)$, $u(t,x,y,z)$)

Ex: (ODE)

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- Malthusian growth model for bacteria

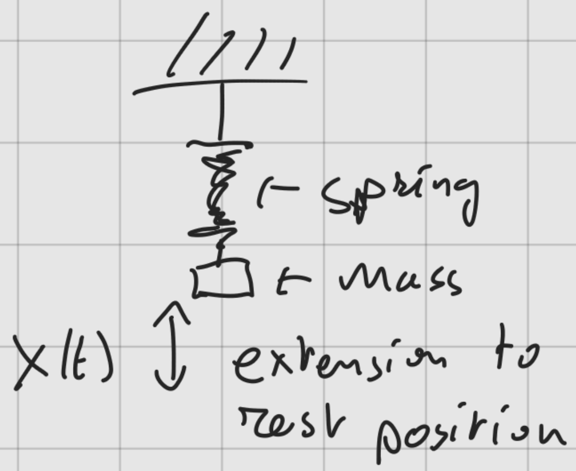
  $\dot{P}(t) = \lambda P(t)$, where

  $\dot{P}(t) = \frac{d}{dt} P(t)$, λ growth parameter (given),

  $P(t) = \#$ bacteria at time t (unknown)

Exact sol. $P(t) = C \cdot e^{\lambda t}$

- Mass-spring system



Newton tells: Mass x Accelaration = Force

$$m \cdot \frac{d^2}{dt^2} x(t) = \underbrace{-k \cdot x(t)}$$

Hooke's law
$k \rightsquigarrow$ stiffness constant (given)

- etc...

To find the exact sol. to ODE $\rightsquigarrow$
need additional conditions

Need additional conditions.

<u>Ex:</u> (IVP)

Adding an initial condition to an ODE gives an <u>initial value problem</u> (<u>IVP</u>):

$$\begin{cases} \dot{P}(t) = \lambda P(t) \\ P(0) = P_0 \end{cases}$$

(IC: says that initial population is given by P_0)

<u>Ex:</u> $\begin{cases} \dot{P}(t) = 6 \cdot P(t) \\ P(0) = 7 \end{cases} \rightarrow P(t) = 7 \cdot e^{6t}$

<u>Ex:</u> (BVP)

Adding boundary conditions (BC) gives <u>Boundary value problems</u> (<u>BVP</u>), f. instance:

$$\begin{cases} -u''(x) = \cos(x) & \text{for } 0 < x < 1 \\ u(0) = 1,\ u(1) = 6 \end{cases}$$

BC → specify sol. at 0 and 1.

<u>Def:</u> <u>Laplace operator</u> is denoted by <u>Δu</u> and reads

1D $\quad \Delta u(x) = u''(x)$

2D $\quad \Delta u(x,y) = \partial^2 u(x,y) + \partial^2 u(x,y) =$

$$= u_{xx} + u_{yy}$$

3D $\quad \Delta u(x,y,z) = u_{xx} + u_{yy} + u_{zz}$

etc..

Rem: Notation $\nabla^2 = \Delta$ physics

Ex: (PDE)

- Laplace equation $\quad \Delta u = 0$

  1D: $u''(x) = 0$ / 2D: $u_{xx}(x,y) + u_{yy}(x,y) = 0$

- Heat equation $\quad u_t - \Delta u = f$

  1D $\quad u_t(x,t) - u_{xx}(x,t) = \sin(t)$

- Wave equation: $\quad u_{tt} - \Delta u = g$

  1D $\quad u_{tt}(x,t) - u_{xx}(x,t) = 5x \cdot t$

Again, one adds conditions in order to specify a unique solution to a PDE.

Ex: (Heat eq. in 1D)

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & \text{for } 0<x<1, t\in[0,T] \quad \text{(DE)} \\ u(0,t) = 0,\ u(1,t) = 6 & \text{for } t\in[0,T] \quad \text{(BC)} \\ u(x,0) = 3x & \text{for } 0<x<1 \quad \text{(IC)} \end{cases}$$

Classification of linear second-order PDE in 2D:

Consider general PDE

$$A\,u_{xx} + B u_{xy} + C u_{yy} + D u_x + E\,u_y + F u = G, \qquad (*)$$

where $u = u(x,y)$ and $A, B, C, \ldots \in \mathbb{R}$ (given)

Define discriminant $\quad d = B^2 - 4AC$

Classification of PDE $(*)$:

Def: The PDE $(*)$ is called

(i) Elliptic if $d < 0$

(ii) Parabolic if $d = 0$

(iii) Hyperbolic if $d > 0$

Ex: (i) Laplace eq: $\Delta u(x,y) = 0$ or $u_{xx} + u_{yy} = 0$

$\Rightarrow$ $A = C = 1$, rest $= 0$: $d = B^2 - 4AC = 0 - 4 = -4$

$\rightsquigarrow$ elliptic problem :-)

(ii) Heat eq. $u_t(x,t) - u_{xx}(x,t) = 0$ "$(x,t) \leftrightarrow (x,y)$ in $(*)$"

$\rightsquigarrow$ parabolic

(iii) Wave eq. $\rightsquigarrow$ hyperbolic type.

Rem: Can do the same if coeff. depends on (x,y) in PDE $(*)$: $A = A(x,y)$, etc.

$\rightsquigarrow$ Then, fix a point (x_0, y_0) and investigate the type of PDE at this point!

Ex: (Book p.5)

Tricomi eq. of gas dynamics : $y \cdot u_{xx} + u_{yy} = 0$

Chapter II: Mathematical tools

Goal: Introduce some (abstract) spaces where we will live and work.

[If $\mathbb{R}^2, \mathbb{R}^3$ is ok → we do the same]

1) Vector space:

Def: A set V of "vectors" or functions is called a vector space (VS) or linear space if $\forall u,v,w \in V$ and $\forall \alpha, \beta \in \mathbb{R}$, one has

(i) $u+(v+w) = (u+v)+w$ (associativity)

(ii) $u+v = v+u$ (commutativity)

(iii) $\exists\, 0 \in V$ s.t. $v+0 = v$ (zero element)

(iv) $\forall v \in V$, $\exists$ inverse $(-v) \in V$ s.t. $v+(-v) = 0$

(v) $\alpha(\beta v) = (\alpha\beta)v$

(vi) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \cdot v = v$

(vii) $\alpha(u+v) = \alpha u + \alpha v$

(viii) $(\alpha+\beta)v = \alpha v + \beta v$

Ex! • $V = \mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$

with $(x_1,y_1) + (x_2,y_2) = (x_1+x_2, y_1+y_2)$

$\alpha(x,y) = (\alpha x, \alpha y)$

satisfy (i)–(viii) → hence V is a VS

• Same for $\mathbb{R}$, $\mathbb{R}^3$, $\mathbb{R}^n$, $(n \in \mathbb{N})$.

• TBC