$\boxed{\text{Recall}}$

- IVP $\to$ ODE + IC
- BVP $\to$ ODE + BC
- PDE $\to$ Laplace eq. $\Delta u = 0$

  Heat eq. $u_t - \Delta u = f$

  Wave eq. $u_{tt} - \Delta u = g$
- Vector space (VS): $\mathbb{R}^n$

Ex: (VS)

- Consider the space of polynomials of degree ≤ 1 defined on $[0,1]$:

  $\mathcal{P}^{(1)}([0,1]) = \{ax+b : a, b \in \mathbb{R}, x \in [0,1]\}$

  is a VS! (with 2 operations:

  $+ \Rightarrow$ says how to add polyn. (what you think)

  $\cdot \to$ multiply with scalar (")

  verify def of VS)

- Similarly, one defines $\mathcal{P}^{(5)}([0,1])$ or

  $\mathcal{P}^{(n)}([c,d])$ for $n \in \mathbb{N}$

  $\mathcal{P}^{(n)}(\mathbb{R})$

Rem: $\leq \leftrightarrow =$: not a VS

for instance: $p_1(x) + p_2(x) = p_3(x)$
$\quad$ deg 1 $\quad$ deg 1 $\quad$ deg 0

Def: Let V be a VS. A subset $U \subset V$ is called a subspace of V if
$$\alpha u + \beta v \in U \quad \forall \alpha, \beta \in \mathbb{R} \; \forall u, v \in U$$

Ex: Consider $V = \mathbb{R}^3$ and $U = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$
show U subspace of V:

(i) $U \subset V$: ok

(ii) Take $u = (u_1, u_2, 0) \in U$ and $v = (v_1, v_2, 0) \in U$ and $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha u + \beta v =$
$$= (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, 0) \in U$$
Def of U.

Ex: Consider $V = \mathcal{P}^{(4)}([0,1])$ and $\mathcal{U} = \mathcal{P}^{(2)}([0,1])$:
Show that $\mathcal{U}$ subspace of V:

(i) $\mathcal{U} \subset V$: clear: poly. of degree ≤ 2 are also polyn. of degree ≤ 4

(ii) $\alpha p_1(x) + \beta p_2(x) \in \mathcal{U}$
deg ≤ 2

Def: Let V be a VS and $v_1, v_2, v_3, \dots, v_n \in V$.
A linear combination of these elements is given by
$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$
for some $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$

Def: Let V be a VS and $v_1, v_2, \dots, v_n \in V$.
The space of all linear combinations of these elements is defined by
$$\mathrm{span}(v_1, v_2, \dots, v_n) = \{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n : \alpha_1, \dots, \alpha_n \in \mathbb{R} \}$$

Ex: Consider the monomials $1, x, x^2$.
What is $\mathrm{span}(1, x, x^2)$?
$$\mathrm{span}(1,x,x^2) \underset{\text{Def span}}{=} \{ a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 : a_0, a_1, a_2 \in \mathbb{R} \} \underset{\text{Def of } \mathcal{P}^{(2)}}{=} \mathcal{P}^{(2)}(\mathbb{R})$$

Let us express the poly. $4x - 5$ in terms of $1, x, x^2$:

"basis"

$$4x - 5 = (-5) \cdot 1 + 4 \cdot x + 0 \cdot x^2$$

"coordinates"

2) Basis of a vector space:

Let V be a VS

Def: • A set $\{v_1, v_2, \dots, v_n\}$ in V is **linearly independent** if the equation

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0$$

has only the trivial sol: $a_1 = a_2 = \dots = a_n = 0$.

• Else, the set is **linearly dependent**

• A set $\{v_1, v_2, \dots, v_n\}$ in V is called **a basis of V** if $\operatorname{span}(v_1, \dots, v_n) = V$ and $\{v_1, \dots, v_n\}$ are linearly independent

• The **dimension of V** is

$\dim(V) = n =$ number of elements in basis

Ex: The $\{1, x, x^2\}$ is a basis of $\mathcal{P}^{(2)}(\mathbb{R})$:

(i) lin. independent:

Consider $a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = \overset{\text{polyn. zero}}{0}$

This eq. can only hold if $a_0 = a_1 = a_2 = 0$.

$\underset{\text{Def}}{\Rightarrow}$ $\{1, x, x^2\}$ is lin. indep.

(ii) $\operatorname{span}(1, x, x^2) = \mathcal{P}^{(2)}(\mathbb{R})$: see above

$\Rightarrow \{1, x, x^2\}$ is a basis of $\mathcal{P}^{(2)}(\mathbb{R})$
and $\dim(\mathcal{P}^{(2)}(\mathbb{R})) = 3$

3) Inner product / scalar product:

Motivation: Scalar prod. in $\mathbb{R}^2$: $(1,2) \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 4 = 11$
generalize this to VS

Let V be a VS.

Def: An inner product / scalar product on V
is a map $(\cdot,\cdot) : V \times V \to \mathbb{R}$ s.t.

(i) $(u,v) = (v,u)$ (symmetry)

(ii) $(u+\alpha v, w) = (u,w) + \alpha (v,w)$ (linearity)

(iii) $(v,v) \geq 0$ (positivity)

(iv) $(v,v) = 0 \Leftrightarrow v = 0$

$\forall \alpha \in \mathbb{R}, \forall u, v, w \in V$

Def: • A VS V with an inner product is called an inner product space. Notation: $(V, (\cdot,\cdot))$
$(V, (\cdot,\cdot)_V)$

• In such spaces, one has a norm defined by

$$\|v\| = \|v\|_V = \sqrt{(v,v)}$$

Ex: $V = \mathbb{R}^2$, $(v,u) = v_1 u_1 + v_2 u_2$

$$\|v\| = \sqrt{(v,v)} = \sqrt{v_1^2 + v_2^2}$$

(euclidean norm)

Def: Let $(V, (\cdot,\cdot))$ be an inner product space. Two elements $u, v \in V$ are orthogonal in V if $(u,v) = 0$. Notation: $u \perp v$.

4) Spaces of integrable functions:

Motivation: $u, v \in \mathbb{R}^n \to (u,v) = \sum_{j=1}^n u_j v_j \to \|u\| = \sqrt{\sum_{j=1}^n u_j^2}$

$\updownarrow$

f, g fct $\to (f,g) = ? \to \|f\| = ?$

Def: Let $\Omega \subset \mathbb{R}^n$ be a domain (connected open set). We define the space of square integrable functions by

$$L^2(\Omega) = \{ \underset{\text{(measurable)}}{f : \Omega \to \mathbb{R}} : \int_\Omega |f(x)|^2 dx < \infty \}$$

(Lebesgue integral)

with the L^2-inner product

$(f,g)_{L^2(\Omega)} = \int_\Omega f(x)\cdot g(x)\,dx$ and norm

$$\|f\|_{L^2(\Omega)} = \sqrt{(f,f)_{L^2(\Omega)}} = \sqrt{\int_\Omega f(x)^2\,dx}$$

Rem: $\mathbb{R}^n$: $(u,v) = \sum_{j=1}^n u_j v_j \qquad \|u\| = \sqrt{\sum_{j=1}^n u_j^2}$

$L^2(0,1)$: $(f,g)_{L^2} = \int_0^1 f(x)g(x)\,dx \qquad \|f\|_{L^2} = \sqrt{\int_0^1 f(x)^2\,dx}$

Ex: Consider 2 polynomials $p(x) = x$ and $q(x) = x^2$ defined on $[-1,1]$. Are they orthogonal in $L^2(-1,1)$?

p and q are orthogonal if $(p,q)_{L^2} = 0$ (Def)

Compute: $(p,q)_{L^2(-1,1)} = \int_{-1}^1 p(x)q(x)\,dx = \int_{-1}^1 x^3\,dx = \ldots = 0$

Def L^2 inner p.

Hence, $p \perp q$ in $L^2(-1,1)$.

We can "generalize" the above def:

Def: • Let $1 \le p < \infty$ ($p\in\mathbb{R}$) and $\Omega \subset \mathbb{R}^n$. We define the space

$L^p(\Omega) = \{ f: \Omega \to \mathbb{R} : \|f\| \ldots < \infty \}$ where

$L^p(\Omega) = \{ f : \Omega \to \mathbb{R} : \|f\|_{L^p(\Omega)} < \infty \}$, where

the L^p-norm reads

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}$$

- For $p = \infty$, one defines

$L^\infty(\Omega) := \{ f : \Omega \to \mathbb{R} : \|f\|_{L^\infty} < \infty \}$ with

the L^∞-norm / sup-norm / max-norm

$$\|f\|_{L^\infty(\Omega)} = \max_{x \in \Omega} |f(x)|$$

[⚠ Not exact definition, but enough for us]

Ex: Compute the L^1-norm of $f(x) = 2x + \sin(x)$ defined on $[0, \frac{\pi}{2}]$:

$$\|f\|_{L^1} \overset{\text{Def}}{=} \int_0^{\pi/2} |f(x)| \, dx = \int_0^{\pi/2} |\underbrace{2x + \sin(x)}_{\geq 0 \text{ on } [0, \pi/2]}| \, dx =$$

$$= \int_0^{\pi/2} (2x + \sin(x)) \, dx = \dots = \pi^2 + 1$$

5) Spaces of differentiable functions:

Def: • For $\Omega \subset \mathbb{R}^n$, we denote $C(\Omega)$ the linear space of continuous functions on Ω.

• The subspace $C_b(\Omega)$ of all bounded fct with maximum norm / sup norm

$$\|f\|_{C(\Omega)} = \sup_{x \in \Omega} |f(x)|$$

Rem: Enough $\Omega \hookrightarrow$ compact set