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Chapter 2: Mathematical tools (summary)

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Goal: Introduce some (abstract) spaces and various mathematical tools and results. This will help us to solve (numerically) differential equations in the next chapters.

- A set V is called a vector space or linear space (VS) if, for all $u, v, w \in V$ and for all $\alpha, \beta \in \mathbb{R}$ one has

  0. $u + \alpha v \in V$ (linearity)
  1. $(u + v) + w = u + (v + w) = u + v + w$ (associativity)
  2. There exists an element $0 \in V$ such that $u + 0 = 0 + u = u$ for all $u \in V$ (zero element)
  3. $u + v = v + u$ (commutativity)
  4. For all $u \in V$, there exists an element $(-u) \in V$ such that $u + (-u) = 0$ (inverse element)
  5. $(\alpha + \beta)u = \alpha u + \beta u$
  6. There exists $1 \in \mathbb{R}$ such that $1v = v$
  7. $\alpha(u + v) = \alpha u + \beta v$
  8. $\alpha(\beta u) = (\alpha\beta)u = \alpha\beta u$
  9. There exists $1 \in \mathbb{R}$ such that $1u = u$ for all $u \in V$.

  The elements in V are called vectors (but they can be something else, like "normal" vectors, matrices, functions, or sequences) and the ones in $\mathbb{R}$ scalars. The above axioms (rules) tell us that we can do anything reasonable with vectors and scalars.

  Example: The vector space of all polynomials, defined on $\mathbb{R}$, of degree $\leq n$ is denoted by

  $$\mathscr{P}^{(n)}(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : \ a_0, a_1, \ldots, a_n \in \mathbb{R}\}.$$

- A subset $U \subset V$ of a VS V is called a subspace of V if $\alpha u + \beta v \in U$ for all $u, v \in U$ and $\alpha, \beta \in \mathbb{R}$.

- Let V be a VS. The space of all linear combinations of the elements $v_1, v_2, \ldots, v_n \in V$ is denoted by

  $$\operatorname{span}(v_1, \ldots, v_n) = \{a_1 v_1 + a_2 v_2 + \ldots + a_n v_n : \ a_1, \ldots, a_n \in \mathbb{R}\}.$$

  Example: $\operatorname{span}(1, x, x^2) = \{a_0 1 + a_1 x + a_2 x^2 : \ a_0, a_1, a_2 \in \mathbb{R}\} = \mathscr{P}^{(2)}(\mathbb{R})$.

- A set $\{v_1, v_2, \ldots, v_n\}$ in a VS V is linearly independent if the equation

  $$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0 \in V$$

  has only the trivial solution $a_1 = a_2 = \ldots = a_n = 0 \in \mathbb{R}$. Else it is called linearly dependent.

  Example: The set $\{1, x, x^2\} \subset \mathscr{P}^{(2)}(\mathbb{R})$ is linearly independent.

- A set $\{v_1, v_2, \ldots, v_n\}$ in a VS V is called a basis of V if the set is linearly independent and $\operatorname{span}(v_1, \ldots, v_n) = V$. The dimension of V is then given by the number of elements of the basis, here $\dim(V) = n$.

  Example: The set $\{1, x, x^2\}$ is a basis of $\mathscr{P}^{(2)}(\mathbb{R})$ and thus $\dim(\mathscr{P}^{(2)}(\mathbb{R})) = 3$.

- A scalar product or inner product on a VS V is a map $(\cdot, \cdot)\colon V \times V \to \mathbb{R}$ such that, for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$,

1. $(u,v) = (v,u)$ (symmetry)
2. $(u+\alpha v, w) = (u,w) + \alpha(v,w)$ (linearity)
3. $(u,u) \geq 0$ (positivity)
4. $(u,u) = 0 \in \mathbb{R}$ if and only if $u = 0 \in V$.

- A VS V with an inner product is called an inner product space, which is denoted by $(V,(\cdot,\cdot))$ or $(V,(\cdot,\cdot)_V)$ or $(V,\langle\cdot,\cdot\rangle_V)$.

  Such space has a norm defined by $\|v\| = \sqrt{(v,v)}$ for all $v \in V$.

- Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v \in V$. u and v are orthogonal if $(u,v) = 0$. Notation: $u \perp v$.

- Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v \in V$. Cauchy–Schwarz inequality (CS) reads

$$|(u,v)| \leq \|u\| \cdot \|v\|.$$

- Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v \in V$. The triangle inequality ($\triangle$) reads

$$\|u+v\| \leq \|u\| + \|v\|.$$

- Example of a VS: The space of square integrable functions defined on the interval $[a,b]$ is denoted by

$$L^2([a,b]) = L^2(a,b) = L_2(a,b) = \left\{ f\colon [a,b] \to \mathbb{R} \colon \int_a^b |f(x)|^2 \,\mathrm{d}x < \infty \right\}.$$

  It is equipped with the inner product

$$(f,g)_{L^2} = \int_a^b f(x)g(x)\,\mathrm{d}x$$

  which induces the norm

$$\|f\|_{L^2} = \sqrt{(f,f)_{L^2}} = \sqrt{\int_a^b |f(x)|^2 \,\mathrm{d}x}.$$

  More generally, for a domain $\Omega \subset \mathbb{R}^n$, one defines

$$L^2(\Omega) = \left\{ f\colon \Omega \to \mathbb{R} \colon \|f\|_{L^2(\Omega)} < \infty \right\},$$

  where $\|f\|_{L^2} = \sqrt{(f,f)_{L^2(\Omega)}}$ and $(f,g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)\,\mathrm{d}x$.

  Similarly, one can also define the spaces $L^p(\Omega)$, for a real number $1 \leq p < \infty$, as well as the space $L^\infty(\Omega)$. These two spaces are equipped with their corresponding norms.

- The space of continuous function defined on $[a,b]$ is given by

$$C^0([a,b]) = \mathscr{C}^0([a,b]) = \mathscr{C}^{(0)}(a,b) = \{f\colon [a,b] \to \mathbb{R} : f \text{ is continuous}\}$$

  and equipped with the norm

$$\|f\|_{C^0([a,b])} = \max_{a \leq x \leq b} |f(x)|.$$

Similarly, for $\Omega \subset \mathbb{R}^n$ a (bounded) domain and k a positive integer, one defines the space of kth continuously differentiable functions

$$C^k(\Omega) = \mathscr{C}^k(\Omega) = \{f\colon \Omega \to \mathbb{R} : D^\alpha f \text{ are continuous for all } |\alpha| \leq k\}$$

and equipped with the norm

$$\|f\|_{C^k(\Omega)} = \sum_{|\alpha|\leq k} \sup_{x\in\Omega} |D^\alpha f(x)|.$$

One shall also use the following space

$$C^k(\overline{\Omega}) = \mathscr{C}^k(\overline{\Omega}) = \left\{f \in C^k(\Omega) : D^\alpha f \text{ can be extended from } \Omega \text{ to its closure } \overline{\Omega}\right\}$$

and equipped with the norm

$$\|f\|_{C^k(\overline{\Omega})} = \sum_{|\alpha|\leq k} \sup_{x\in\overline{\Omega}} |D^\alpha f(x)|.$$

- For a positive integer k and $\Omega \subset \mathbb{R}^n$ open, one considers the Sobolev space

$$H^k(\Omega) = \{f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \text{ for } |\alpha| \leq k\}$$

with the inner product

$$(f,g)_{H^k} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha f(x) D^\alpha g(x)\,\mathrm{d}x$$

and norm

$$\|f\|_{H^k} = \sqrt{(f,f)_{H^k}}.$$

For $k=1$ and $n=1$ and in dimension one, the above norm reads

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|f'\|_{L^2}^2.$$

- The triangle inequality as well as Cauchy–Schwarz can be extended to L^p spaces.

Minkowski's inequality: Consider a domain $\Omega \subset \mathbb{R}^n$, $1 \leq p \leq \infty$ and $f, g \in L^p(\Omega)$. One then has

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Hölder's inequality: Consider a domain $\Omega \subset \mathbb{R}^n$, $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, and $g \in L^q(\Omega)$. One then has

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

This is Cauchy–Schwarz for $p = q = 2$.

- Poincaré inequality ($1d$): Let $L > 0$ and consider the open interval $\Omega = (0, L)$. One then has

$$\|u\|_{L^2(\Omega)} \leq \frac{L}{\sqrt{2}} \|u'\|_{L^2(\Omega)}$$

for all $u \in H_0^1 = \{v \in H^1(\Omega) : v(0) = 0, v(L) = 0\}$.

- Trace theorem ($p = 2$): Let $\Omega \subset \mathbb{R}^n$ (bounded domain with Lipschitz boundary). One then has

$$\|u\|_{L^2(\partial\Omega)}^2 \leq C \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}$$

for all $u \in H^1(\Omega)$.

- The strong form of Poisson's equation in dimension one reads

$$\begin{cases} -u''(x) = f(x) & \text{for } x \in \Omega = (0,1) \\ u(0) = 0, u(1) = 0, \end{cases}$$

  where $f: \Omega \to \mathbb{R}$ is a given function (bounded and continuous for instance).

  The weak form or variational formulation (VF) reads

$$\text{Find } u \in H_0^1(\Omega) \text{ s.t. } (u', v')_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \text{ for all } v \in H_0^1(\Omega).$$

  The minimisation problem (MP) reads

$$\text{Find } u \in H_0^1(\Omega) \text{ s.t. } F(u) \text{ is minimal,}$$

  where the functional $F: H_0^1(\Omega) \to \mathbb{R}$ is defined by $F(v) = \frac{1}{2}(v', v')_{L^2(\Omega)} - (f, v)_{L^2(\Omega)}$ for $v \in H_0^1(\Omega)$.

  We have proved that

$$\text{Strong} \Longrightarrow \text{VF} \Longleftrightarrow \text{MP}$$

  and if in addition $u \in C^2(\Omega)$

$$\text{Strong} \Longleftarrow \text{VF}.$$

- Lax–Milgram theorem: Consider a Hilbert space H, a bounded and coercive bilinear form $a: H \times H \to \mathbb{R}$, and a bounded linear functional $\ell: H \to \mathbb{R}$. Then, there exists a unique element $u \in H$ solution to the equation

$$a(u, v) = \ell(v) \quad \text{for all } v \in H.$$

  Lax–Milgram's theorem can be used, for instance, to find a unique solution to the VF of Poisson's equation seen above.

Further resources:

- wikipedia.se VS
- wikipedia.se InnerProduct
- wikipedia.se Lp spaces
- wikipedia.se CS
- VS and InnerProduct
- function spaces by Terry Tao
- Sobolev spaces and PDE (a little bit more advanced)
- application and proof of LM (more advanced)