R · Vector spaces (VS):

$\mathbb{R}^n$, scalar prod. $x \cdot y = x^T y$ + norm

$\mathcal{P}^{(3)}([4,5])$ poly. deg ≤ 3

$L^2(\Omega)$ with inner prod. and norm

$$(f,g)_{L^2} = \int_\Omega f(x)\, g(x)\, dx$$

$$\|f\|_{L^2} = \sqrt{(f,f)_{L^2}} = \sqrt{\int_\Omega |f(x)|^2\, dx}$$

$\mathcal{C}(\Omega)$ + norm $\|f\|_{\mathcal{C}} = \sup_{x \in \Omega} |f(x)|$

Think: $\Omega \leftrightarrow [0,1]$ or $(0,1)$

Def: · Let $\Omega \subset \mathbb{R}^n$ domain, for any integer $k \geq 0$, denote $\mathcal{C}^k(\Omega)$ the space of all fct that are k-times continuously differentiable

· Define:

$\mathcal{C}^k(\bar{\Omega}) = \{ f \in \mathcal{C}^k(\Omega) : D^\alpha f$ can be extended from Ω to its closure $\bar{\Omega}$ for all multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $|\alpha| \leq k \}$.

We equip the space $\mathcal{C}^k(\bar{\Omega})$ with the

We equip the space $C^k(\bar{\Omega})$ with the supremum norm

$$\|f\|_{C^k(\bar{\Omega})} := \sum_{|\alpha| \leq k} \sup_{x \in \bar{\Omega}} |D^\alpha f(x)|$$

Rem:
- $\bar{\Omega}$ = closure of Ω = smallest closed set that contains Ω
- $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_j \in \mathbb{N}$

  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$
- $$D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}}$$

Ex: Consider $n = 1$, $\Omega = [3, 4]$, $k = 1$. Then,

$$\|f\|_{C^k(\bar{\Omega})} \overset{\text{Def}}{=} \sum_{|\alpha| \leq 1} \sup_{x \in [3,4]} |D^\alpha f(x)| =$$

$$= \sup_{x \in [3,4]} |f(x)| + \sup_{x \in [3,4]} |f'(x)|$$

Ex: Consider $n = 1$, $\Omega = (1, 2)$. Hence, $\bar{\Omega} = [1, 2]$.

The function

$$f(x) = \frac{1}{x - 1}$$

belongs to $\mathcal{C}^k(\Omega)$ $\forall k$ but not to $\mathcal{C}^k(\bar{\Omega})$ since

f and all its derivatives are not defined at $x=1$

6) Sobolev spaces:

Let $\Omega \subset \mathbb{R}^n$ open, $k \geq 0$ integer, and $p \in [1,\infty]$.

Def: • The Sobolev space of order k is defined by

$$W^{k,p}(\Omega) = \{ f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for } |\alpha| \leq k \}$$

[weak derivative, distributional deriv.]

with norm

$$\|f\|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \Big)^{1/p} \quad \text{for } 1 \leq p < \infty$$

$$\|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)} .$$

• We shall also consider the semi-norm

$$|f|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha| = k} \|D^\alpha f\|^p_{L^p(\Omega)} \Big)^{1/p} \quad \text{for } 1 \leq p < \infty$$

$$|f|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| = k} \|D^\alpha f\|_{L^\infty(\Omega)} .$$

• For $p=2$, we denote $W^{k,2}(\Omega) = H^k(\Omega)$,

where

⚠ $H^k(\Omega) = \{ f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \text{ for } |\alpha| \le k \}$

with inner product

$$(f,g)_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha f(x)\, D^\alpha g(x)\, dx$$

$(n=2)$

$\alpha = (\alpha_1, \alpha_2)$, $\alpha_j \in \mathbb{N}$

$|\alpha| = \alpha_1 + \alpha_2$

$k=1 \rightsquigarrow |\alpha| \le k \Rightarrow |\alpha| \le 1 \Rightarrow |\alpha| = 0$ or $|\alpha| = 1$

$\Downarrow$

$\alpha = (0,0)$

$\alpha = (1,0)$ or $\alpha = (0,1)$

Rem: $H^1([a,b]) \subset \mathcal{C}([a,b])$ (ok in dim. 1)

7) Important inequalities:

Def/Th: Let $\Omega \subset \mathbb{R}^n$ domain and $1 \le p \le \infty$, then for all $f, g \in L^p(\Omega)$, one has

Minkowski inequality: $\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$

[Triangle ineq. in $L^p(\Omega)$]

Def / Th: Let $p, q \in [1, \infty]$ that are conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, one has

Hölder's inequality: $\|f \cdot g\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \cdot \|g\|_{L^q(\Omega)}$

$$\|f \cdot g\|_{L^1(\Omega)} = \int_\Omega |f(x) \cdot g(x)| \, dx$$

i.e. $\int_\Omega |f(x) \cdot g(x)| \, dx$

[Ex: $p = q = 2 \rightsquigarrow$ Cauchy-Schwarz ineq. (C-S)]

The following result will be used when studying linear BVP/PDE:

Th (Poincaré inequality)

Let $L > 0$ and $\Omega = (0, L)$. Then, one has

$$\|u\|_{L^2(\Omega)} \leq \underbrace{C_L}_{\text{constant depends on } L} \cdot \|u'\|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega),$$

where $H_0^1(\Omega) = \{u : \Omega \to \mathbb{R} : \underbrace{u \in H^1(\Omega)}_{[u, u' \in L^2(\Omega)]} \text{ and } u(0) = u(L) = 0\}$

Proof: • Fundamental theorem of calculus:

$$u(x) = \underbrace{u(0)}_{= 0 \text{ since } u \in H_0^1} + \int_0^x u'(\xi) \, d\xi \quad \forall x \in \Omega$$

$$\Rightarrow |u(x)|^2 = \left| \int_0^x 1 \cdot u'(\xi) d\xi \right|^2 \leq \left(\int_0^x 1^2 d\xi \right) \cdot \left(\int_0^x |u'(\xi)|^2 d\xi \right)$$

$$\leq \underset{\substack{\uparrow \\ |x| \leq L \text{ since } x \in \Omega}}{x} \cdot \int_0^L |u'(\xi)|^2 d\xi$$

- Integrate above $\int_0^L$:

$$\underbrace{\int_0^L |u(x)|^2 dx}_{\|u\|^2_{L^2(\Omega)}} \leq \int_0^L x \cdot \underbrace{\int_0^L |u'(\xi)|^2 d\xi}_{\|u'\|^2_{L^2(\Omega)}} \, dx$$

That is: $\|u\|^2_{L^2} \leq \underbrace{\int_0^L x \, dx}_{\frac{L^2}{2}} \cdot \|u'\|^2_{L^2}$

or $\|u\|_{L^2} \leq \underset{\substack{\uparrow \\ \text{const. depends on } L}}{C_L} \cdot \|u'\|_{L^2}$ $\blacksquare$

Th: (Trace theorem)

Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ nice domain (bounded domain + Lipschitz boundary).

For $u \in W^{1,p}(\Omega)$, one has

$$\|u\|_{L^p(\partial\Omega)} \leq C \cdot \|u\|^{1-1/p}_{L^p(\Omega)} \cdot \|u\|^{1/p}_{W^{1,p}(\Omega)}$$

For $p=2$, this reads

$$\|u\|^2_{L^2(\partial\Omega)} \leq C \cdot \|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)}$$

Finally, we will use

Grönwall's lemma:

Let $\alpha, \beta, u : [0,\infty) \longrightarrow \mathbb{R}$ continuous with

$$u(t) \leq \alpha(t) + \int_0^t \beta(s) u(s)\, ds \qquad \forall\, t \geq 0$$

a) If β is non-negative, then

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\Big(\int_s^t \beta(r)\, dr\Big)\, ds \qquad \forall t \geq 0$$

b) If, in addition, α is non-decreasing, then

$$u(t) \leq \alpha(t) \cdot \exp\Big(\int_0^t \beta(s)\, ds\Big) \qquad \forall\, t \geq 0.$$

8) Strong form, weak/variational form and minimization:

Consider Poisson's equation

$$(\text{BVP}) \begin{cases} -u''(x) = f(x) & \text{for } x \in \Omega = (0,1) \\ u(0) = 0, \; u(1) = 0, \end{cases}$$

where f is given ...

where f is give data (bounded, cont.)

Aim: Find u $(\in C^2(\Omega))$

The above formulation of the problem is called the <u>strong form</u> of the problem.

next chapter.

To derive a <u>finite element method</u> (FEM), one uses a reformulation of (BVP).

We now present this (equivalent) formulation.

We follow an idea from Galerkin:

Consider $H_0^1(\Omega) = \{ u \in H^1(\Omega) : u(0)=0, u(1)=0 \}$

Multiply (BVP) with a test function $v \in H_0^1$, then integrate, by parts:

$$\underbrace{-\int_0^1 u''(x)\,v(x)\,dx}_{} = \int_0^1 f(x)\,v(x)\,dx = (f,v)_{L^2}$$

$$\underbrace{-u'(x)v(x)\Big|_0^1}_{-u'(1)v(1)+u'(0)v(0)=0 \text{ since } v\in H_0^1} + \underbrace{\int_0^1 u'(x)\,v'(x)\,dx}_{(u',v')_{L^2}}$$

$\hookrightarrow$ We get the eq. $(u',v')_{L^2} = (f,v)_{L^2} \quad \forall v \in H_0^1$

We thus obtain the weak formulation/

Variational formulation of the problem:

(VF) Find $u \in H_0^1$ s.t. $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in H_0^1$

$$\Updownarrow$$

$$\int_0^1 u'(x) v'(x) \, dx = \int_0^1 f(x) v(x) \, dx$$

To get yet another reformulation of (BVP), we follow an idea of Ritz:

Consider the functional

$$F(v) := \frac{1}{2} (v', v')_{L^2} - (f, v)_{L^2} \quad \forall v \in H_0^1.$$

To see why F is important, consider u solution to (VF) and compute $F(u+v)$