R

(BVP) $\begin{cases} -u''(x) = f(x) & x \in \Omega = (0,1) \\ u(0) = 0, \; u(1) = 0 \end{cases}$ STRONG

(VF) Find $u \in H_0^1$ s.t. $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in H_0^1$

WEAK

where $H_0^1 = \{ u \in H^1 \text{ s.t. } u(0) = u(1) = 0 \}$

$$(f, g)_{L^2} = \int_0^1 f(x) g(x) \, dx$$

$$F(v) := \frac{1}{2} (v', v')_{L^2} - (f, v)_{L^2} \qquad \forall v \in H_0^1$$

Quizz + feedback

To see why F is important, take u to be sol. to (VF) and any $v \in H_0^1$ and compute

prop. of inner p.

$$F(u+v) \underset{\text{Def } F}{=} \frac{1}{2} (u'+v', u'+v')_{L^2} - (f, u+v)_{L^2} =$$

$$= \frac{1}{2}(u',u')_{L^2} + \underbrace{(u',v')_{L^2}}_{\downarrow} + \underbrace{\frac{1}{2}(v',v')_{L^2}}_{\geq 0}$$

$$-(f,u)_{L^2} - \underbrace{(f,v)_{L^2}}_{\downarrow}$$

$=0$ since u sol. to (VF)

$$\geq \frac{1}{2}(u',u')_{L^2} - (f,u)_{L^2} = F(u)$$

Def F

$\Leftrightarrow \quad F(u+v) \geq F(u) \quad \forall\, v \in H_0^1 \quad$ (if u sol. (VF))

That is, we obtain the minimization prob:

(MP) Find $u \in H_0^1$ s.t. $F(u)$ is minimal

Break: u sol. to (BVP) (STRONG) $\Rightarrow$ u sol. (VF) (WEAK) $\Rightarrow$ u sol. (MP)

Reverse ?

"(MP) $\Rightarrow$ (VF)":

Take u sol. to (MP), this implies that

$$F(u) \leq F(u+\varepsilon v) \quad \forall \varepsilon \in \mathbb{R},\ \forall v \in H_0^1$$

Idea of Euler: Consider $F(u+\varepsilon v)$ as a fct of ε.

Since u minimize F, then $\frac{\partial}{\partial \varepsilon} F(u+\varepsilon v) \Big|_{\varepsilon=0} = 0$

$$F(u+\varepsilon v) \overset{\text{Def } F}{=} \frac{1}{2}(u'+\varepsilon v', u'+\varepsilon v')_{L^2} - (f, u+\varepsilon v)_{L^2} \overset{\text{prop. of inner p.}}{=}$$

$$= \frac{1}{2}(u', u')_{L^2} + \varepsilon(u', v')_{L^2} + \frac{1}{2}\varepsilon^2 (v', v')_{L^2}$$

$$- (f, u)_{L^2} - \varepsilon (f, v)_{L^2}.$$

Then, we get

$$0 = \frac{\partial}{\partial \varepsilon} F(u+\varepsilon v)\Big|_{\varepsilon=0} = (u', v')_{L^2} - (f, v)_{L^2}$$

Hence, $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall H_0^1$

That is u sol. to (VF).

<u>(VF) $\Rightarrow$ (BVP)</u>: Assume in addition that $u \in \mathcal{C}^2(\Omega)$.

From (VF), we get $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in H_0^1$

Do an integrat. by parts, and get:

$$-\int_0^1 u''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx \qquad [\text{using } v \in H_0^1]$$

That

$$\int_0^1 (u''(x) + f(x))\, v(x)\,dx = 0 \qquad \forall v \in H_0^1 \qquad (*)$$

We now show, by contradiction, that $u''+f$ must be equal to zero on $\Omega = (0,1)$.

Assume that $u''+f \neq 0$, then, by continuity of u'' and f, there must be an $x \in \Omega$ and a neighborhood of x s.t. $u''+f \neq 0$ in this neighborhood.

Now, choose $v \in H_0^1$ with same sign as $u''+f$ in this neighborhood and zero outside.

We than obtain, that

$$\int_0^1 (u''(x) + f(x)) \cdot v(x)\,dx > 0 \implies \Leftarrow$$

contradict (*)

In conclusion, we must have $u''+f \equiv 0$ in Ω
that is $-u''(x) = f(x)$ in Ω, that is (BVP).

$\hookrightarrow$

<u>Th!</u> Strong $\Rightarrow$ weak / VF $\Leftrightarrow$ MP
$\Leftarrow \oplus\ u \in \mathcal{C}^2(\Omega)$

? Does (VF) has unique sol. ???
[also OK for other problems]

9) Lax-Milgram theorem:

<u>Recall:</u> (BVP) $\begin{cases} -u''(x) = f(x) & \text{in } (0,1) \\ u(0) = 0 = u(1) \end{cases}$

" $\Updownarrow$ "

(VF) Find $\underbrace{u \in H_0^1}_{\text{Goal: general space}}$ s.t. $\underbrace{(u', v')_{L^2}}_{\text{bilinear form}} = \underbrace{(f, v)_{L^2}}_{\text{linear form}}$ $\forall v \in H_0^1$

<u>Generalisation:</u>

Consider the problem:

Find $u \in H$ s.t. $a(u,v) = \ell(v)$ $\forall v \in H$,

where

- H is a Hilbert space, that is

  a complete inner product space

  (all Cauchy seq. converge in H)

  Ex: $\mathbb{R}^n$ + Euclidean inner prod.

  L^2 + L^2-inner prod.

  H^1 + H^1-inner prod.

  H^1_0 + H^1-inner prod (used above for Poisson's eq.)

- $\ell : H \to \mathbb{R}$ a bounded linear functional

  that is $|\ell(v)| \leq C \cdot \|v\|_H$ $\forall v \in H$

  (e.g. Poisson: $\ell(v) = (f, v)_{L^2}$)

- $a : H \times H \to \mathbb{R}$ a bounded coercive bilinear form

  (e.g. Poisson $a(u,v) := (u', v')_{L^2}$)

that is:

bilinear: $a(\lambda u + \gamma v, w) = \lambda a(u,w) + \gamma a(v,w)$

$a(u, \lambda v + \gamma w) = \lambda a(u,v) + \gamma a(u,w)$
$\forall \lambda, \gamma \in \mathbb{R}, \forall u, v, w \in H.$

bounded: $\exists \alpha > 0$ s.t. $|a(u,v)| \leq \alpha \cdot \|u\|_H \cdot \|v\|_H$
$\forall u, v \in H.$

coercive: $\exists \beta > 0$ s.t. $a(u,u) \geq \beta \cdot \|u\|_H^2 \quad \forall u \in H$

Th: (Lax-Milgram (LM))
Let a and ℓ be as above.
Then $\exists!$ element $u \in H$ s.t.
$$a(u,v) = \ell(v) \quad \forall v \in H.$$

We now apply (LM) to show that the (VF) of Poisson's eq. has a unique sol. in H_0^1.

Th: Let $f \in L^2(0,1)$. Then $\exists!$ $u \in H_0^1(0,1)$ s.t.
$$(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in H_0^1.$$

Proof:
First set $a(u,v) \overset{\text{Def}}{:=} (u', v')_{L^2}$
$\ell(v) := (f, v)$ $\quad H = H'$

$$\ell(v) := (f, v)_{L^2} \qquad f' = f|_0$$

Def

(i) ℓ is a bounded linear functional:

linearity: ok since we have an inner prod. which is linear.

bounded: $|\ell(v)| \overset{\text{Def } \ell}{=} |(f,v)_{L^2}| \overset{\text{C-S}}{\leq} \underbrace{\|f\|_{L^2}}_{C_f \text{ since } f \in L^2} \cdot \|v\|_{L^2}$

$\leq C_f \cdot \|v\|_{L^2} \leq C_f \cdot \|v\|_{H^1}$ ok

Def norm $\|v\|_{H^1}^2 = \|v\|_{L^2}^2 + \|v'\|_{L^2}^2$

(ii) a is bilinear: Ok since $(\cdot,\cdot)_{L^2}$ is linear derivatives linear

(iii) a is bounded:

$|a(u,v)| \overset{\text{Def } a}{=} |(u',v')_{L^2}| \overset{\text{C-S}}{\leq} \|u'\|_{L^2} \cdot \|v'\|_{L^2} \leq$

$\overset{\text{Def } \|\cdot\|_{H^1}}{\leq} 1 \cdot \|u\|_{H^1} \cdot \|v\|_{H^1} \Rightarrow$ Ok with $\alpha = 1$

(iv) $a(\cdot,\cdot)$ is coercive:

Poincaré : $\|u\|_{L^2} \le \frac{1}{\sqrt{2}} \|u'\|_{L^2} \quad \forall u \in H_0^1$

We consider

$$\|u\|_{H^1}^2 \overset{\text{Def norm}}{=} \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 \overset{\text{Poincaré}}{\le} \frac{1}{2}\|u'\|_{L^2}^2 + \|u'\|_{L^2}^2$$

$$\le \frac{3}{2} \cdot \|u'\|_{L^2}^2 \overset{\text{Def } \|\cdot\|_{L^2}}{=} \frac{3}{2} (u', u')_{L^2} \overset{\text{Def of } a}{=} \frac{3}{2} a(u,u)$$

$\Rightarrow a(u,u) \ge \underbrace{\frac{2}{3}}_{=\beta} \|u\|_{H^1}^2 \quad \Rightarrow a(\cdot,\cdot)$ coercive OK in H_0^1

$\Rightarrow$ Apply LM and get $\exists!$ to (VF) in H_0^1 $\blacksquare$