Chapter 3: A Galerkin FEM for BVP (summary)

January 24, 2022

Goal: Provide an introduction to finite element methods (FEM) for BVP.

- Let m be a positive integer. Denote a partition of the interval $[0,1]$ into $m+1$ subintervals by $\tau_h : 0 = x_0 < x_1 < \ldots < x_m < x_{m+1} = 1$, where $h_j = x_j - x_{j-1}$ for $j = 1, 2, \ldots, m+1$ (we shall mainly consider the case of a uniform partition, where $h_j = h$ constant). We define the hat function $\{\varphi_j\}_{j=0}^{m+1}$ by

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h_j} & \text{for } x_{j-1} \leq x \leq x_j \\ \frac{x - x_{j+1}}{-h_{j+1}} & \text{for } x_j \leq x \leq x_{j+1} \\ 0 & \text{else} \end{cases}$$

  for $j = 1, \ldots, m$. The functions $\varphi_0(x)$ and $\varphi_{m+1}(x)$ are defined as half hat functions.

  With the above, one then defines the space of continuous piecewise linear functions on $[0,1]$ by

$$V_h = V_h(0,1) = \operatorname{span}\left(\varphi_0, \varphi_1, \ldots, \varphi_{m+1}\right) = \left\{v \colon [0,1] \to \mathbb{R} \quad : \ v \text{ cont. piecewise linear on } \tau_h\right\}.$$

  As usual, one has $v(x) = \sum_{j=0}^{m+1} \zeta_j \varphi_j(x)$, where $\zeta_j = v(x_j)$, for any $v \in V_h$.

- In a nutshell, a Galerkin finite element method (FEM) for the BVP with homogeneous Dirichlet BC

$$\begin{cases} -u''(x) = f(x) & \text{for} \quad x \in (0,1) \\ u(0) = 0, u(1) = 0 \end{cases}$$

  consists of the following

  1. Multiply the DE by a test function $v \in H_0^1 = \left\{v \colon [0,1] \to \mathbb{R} : \ v, v' \in L^2(0,1) \text{ and } v(0) = v(1) = 0\right\}$. Integrate the above over the domain $[0,1]$ and get the variational formulation of the problem (VF)

$$\text{Find} \quad u \in H_0^1 \quad \text{such that} \quad \int_0^1 u'(x) v'(x) \, \mathrm{d}x = \int_0^1 f(x) v(x) \, \mathrm{d}x \quad \text{for all} \quad v \in H_0^1$$

     or shortly

$$\text{Find} \quad u \in H_0^1 \quad \text{such that} \quad (u', v')_{L^2(0,1)} = (f, v)_{L^2(0,1)} \quad \forall v \in H_0^1.$$

  2. Specify the finite dimensional space $V_h^0 \subset H_0^1$ defined as $V_h^0 = \operatorname{span}(\varphi_1, \ldots, \varphi_m)$, for the above hat functions φ_j. Consider the FE problem

$$\text{Find} \quad u_h \in V_h^0 \quad \text{such that} \quad (u_h', v_h')_{L^2(0,1)} = (f, v_h)_{L^2(0,1)} \quad \forall v_h \in V_h^0.$$

  3. Insert the ansatz

$$u_h(x) = \sum_{j=1}^{m} \zeta_j \varphi_j(x)$$

     into the FE problem and take $v_h = \varphi_i$, for $i = 1, \ldots, m$, to get a linear system of equation for the unknown $\zeta = (\zeta_1, \ldots, \zeta_m)$:

$$S\zeta = b.$$

     Here, $S = (s_{i,j})_{i,j=1}^m$ is termed the stiffness matrix (with entries $s_{ij} = (\varphi_i', \varphi_j')_{L^2(0,1)}$) and $b = (b_i)_{i=1}^m$ the load vector (with entries $b_i = (f, \varphi_i)_{L^2(0,1)}$).

Further resources:

- FE at wikiversity.org
- FE at github.io
- FEM course notes at web.stanford.edu
- FEM for BVP at amath.unc.edu
- FEM by Gilbert Strang on youtube, good!