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Chapter 4: Interpolation and numerical integration (summary)

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Goal: Interpolation: We want to pass a (simple) function through a given set of data points.
Numerical integration: We want to find numerical approximations of integrals $\int_a^b f(x)\,\mathrm{d}x$.

- Let q be a positive integer. Consider an interval $[a,b]$ and a grid of $(q+1)$ distinct points $x_0 = a < x_1 < \ldots < x_q = b$. One defines Lagrange polynomials by

$$\lambda_i(x) = \prod_{j=0, j\neq i}^{q} \frac{x - x_j}{x_i - x_j}$$

  for $i = 0, 1, \ldots, q$. One then has

$$\mathscr{P}^{(q)}(a,b) = \operatorname{span}\big(\lambda_0(x), \lambda_1(x), \ldots, \lambda_q(x)\big),$$

  where we recall that $\mathscr{P}^{(q)}(a,b)$ denotes the set of polynomials, defined on $[a,b]$, of degree at most q.

- Let $q \in \mathbb{N}$. Consider a continuous function $f\colon [a,b] \to \mathbb{R}$ and $q+1$ distinct interpolation points $\big(x_j, f(x_j)\big)_{j=0}^{q}$ with $a = x_0 < x_1 < \ldots < x_q = b$. A polynomial $\pi_q f \in \mathscr{P}^{(q)}(a,b)$ is a polynomial interpolant for f if

$$\pi_q f(x_j) = f(x_j) \quad \text{for} \quad j = 0, 1, 2, \ldots, q.$$

  Examples of polynomial interpolants: Remembering that $\mathscr{P}^{(q)}(a,b) = \operatorname{span}(1, x, x^2, \ldots, x^q)$, one gets a polynomial interpolant $\pi_q f$ in the standard basis. Taking $\mathscr{P}^{(q)}(a,b) = \operatorname{span}(\lambda_0(x), \lambda_1(x), \ldots, \lambda_q(x))$, one gets the Lagrange interpolant $\pi_q f$. These are the same polynomials (seen in a different basis).

- Let m be a positive integer. Consider a uniform partition of an interval $[a,b]$, denoted $\tau_h : x_0 = a < x_1 < \ldots < b = x_{m+1}$ with $h = x_j - x_{j-1}$, and the space of continuous piecewise linear functions on τ_h, $V_h = \operatorname{span}(\varphi_0, \ldots, \varphi_{m+1})$. Define the mesh function $h(x) = h_j$ if $x \in (x_{j-1}, x_j)$ and $j = 1, 2, \ldots, m+1$.

  The continuous piecewise linear interpolant of f is defined by

$$\pi_h f(x) = \sum_{j=0}^{m+1} f(x_j)\varphi_j(x) \quad \text{for} \quad x \in [a,b].$$

  If $f \in \mathscr{C}^2(a,b)$ (can be relaxed) one has, for instance, the following bounds for the interpolation error for the continuous piecewise linear interpolant on a uniform partition

$$\begin{aligned}
\|\pi_h f - f\|_{L^p(a,b)} &\leq Ch^2 \|f''\|_{L^p(a,b)}, \\
\|\pi_h f - f\|_{L^p(a,b)} &\leq Ch \|f'\|_{L^p(a,b)}, \\
\|(\pi_h f)' - f'\|_{L^p(a,b)} &\leq Ch \|f''\|_{L^p(a,b)},
\end{aligned}$$

  for $p = 1, 2, \infty$.

  In case of non-uniform partitions, one uses a mesh function $h(x)$ and gets for instance

$$\|\pi_h f - f\|_{L^p(a,b)} \leq C \|h^2 f''\|_{L^p(a,b)}$$

  and similarly for the other estimates.

- Let us give 3 classical quadrature rules to numerically approximate the integral $\int_a^b f(x)\,\mathrm{d}x$:

  The midpoint rule reads
  $$\int_a^b f(x)\,\mathrm{d}x \approx (b-a) f\left(\frac{a+b}{2}\right).$$
  The trapezoidal rule reads
  $$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{2}\big(f(a)+f(b)\big).$$
  The Simpson rule reads
  $$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right).$$

  In practice, one first considers a (uniform) partition of the interval $[a,b]$, $a = x_0 < x_1 < \ldots < x_N = b$, and then apply a quadrature rule (denoted by $QF(x_j, x_{j+1}, f)$ below) on each small subintervals:
  $$\int_a^b f(x)\,\mathrm{d}x = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x)\,\mathrm{d}x \approx \sum_{j=0}^{N-1} QF(x_j, x_{j+1}, f).$$

Further resources:

- Lagrange interpolation on youtube
- Lagrange interpolation on www.dcode.fr
- interpolation at www.maths.lth.se
- Lagrange interpolation at www.phys.libretexts.org
- trapezoidal rule at www.khanacademy.org
- quadrature formulas at tutorial.math.lamar.edu