2) Consider the IVP

$$(*)\quad \begin{cases} \dot u(t) = \lambda u(t), & 0 < t \le 1 \\ u(0) = u_0 \end{cases}$$

Compute the Galerkin approximation for $q = 1,2,3,4$ for $\lambda = u_0 = 1$.

<u>Solution</u>

VF: Find $u \in \mathcal{P}^{(q)}(0,1)$ s.t.

$$\int_0^1 \dot u(t)\, v(t) = \int_0^1 \lambda u(t)\, v(t) \qquad \forall\, v \in V_0^{(q)} := \left\{ v \in \mathcal{P}^{(q)}(0,1) : v(0) = 0 \right\}$$

Ansatz: $$u(t) = \sum_{j=0}^{q} \xi_j t^j = \{u(0) = u_0\} = u_0 + \sum_{j=1}^{q} \xi_j t^j$$

$$\dot u(t) = \sum_{j=1}^{q} j \xi_j t^{j-1}$$

$\{t^i\}_{i=1}^{q}$ spans the test space $V_0^{(q)}$

$\Rightarrow$ Find ξ_j, $j = 1, \dots, q$ so that

$$\int_0^1 \sum_{j=1}^{q} j \xi_j t^{j-1} \cdot t^i = \int_0^1 \left(\lambda u_0 + \lambda \sum_{j=1}^{q} \xi_j t^j \right) t^i , \qquad i = 1, \dots, q$$

$$\sum_{j=1}^{q} \xi_j \underbrace{\left(j \int_0^1 t^{i+j-1} - \lambda \int_0^1 t^{i+j} \right)}_{a_{ij}} = \underbrace{\lambda u_0 \int_0^1 t^i}_{b_i} \Rightarrow A\xi = b$$

$$\left.\begin{aligned} \int_0^1 t^{i+j-1} &= \left[\frac{t^{i+j}}{i+j} \right]_0^1 = \frac{1}{i+j} \\ \int_0^1 t^{i+j} &= \left[\frac{t^{i+j+1}}{i+j+1} \right]_0^1 = \frac{1}{i+j+1} \end{aligned}\right\} \Rightarrow a_{ij} = \frac{j}{i+j} - \frac{\lambda}{i+j+1}$$

$$\int_0^1 t^i = \left[\frac{t^{i+1}}{i+1}\right]_0^1 = \frac{1}{i+1} \Rightarrow b_i = \frac{\lambda u_0}{i+1}$$

$\underline{q=1}$ $\quad a_{11} = \ldots = \frac{1}{6} \qquad b_1 = \frac{1}{2}$

$\Rightarrow$ our linear system is $\frac{1}{6}\xi_1 = \frac{1}{2} \Rightarrow \xi_1 = 3$

$\Rightarrow \underline{u_1(t) = 1 + 3t}$

$\underline{q=2}$ $\quad A = \ldots = \begin{bmatrix} \frac{1}{6} & \frac{5}{12} \\ \frac{1}{12} & \frac{3}{10} \end{bmatrix}, \quad b = [1/2 \quad 1/3]^T$

$A\xi = b \Rightarrow \xi = \begin{bmatrix} 8/11 \\ 10/11 \end{bmatrix}$

$\Rightarrow \underline{u_2(t) = 1 + \frac{8}{11}t + \frac{10}{11}t^2}$

$\underline{q=3}$ $\quad A = \ldots = \begin{bmatrix} \frac{1}{6} & \frac{5}{12} & \frac{11}{20} \\ \frac{1}{12} & \frac{3}{10} & \frac{13}{30} \\ \frac{1}{20} & \frac{7}{30} & \frac{5}{14} \end{bmatrix} \qquad b = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$

$A\xi = b \Rightarrow \xi = \begin{bmatrix} \frac{30}{29} & \frac{45}{116} & \frac{35}{116} \end{bmatrix}^T$

$\underline{u_3(t) = 1 + \frac{30}{29}t + \frac{45}{116}t^2 + \frac{35}{116}t^3}$

3) Compute the $L^2(0,1)$-projection into $\mathcal{P}^3(0,1)$ of the exact solution u for $(*)$ with $\lambda = u_0 = 1$ and compare with the Galerkin solution.

Solution

Exact solution: $u = u_0 e^{\lambda t} = e^t$.

The $L^2(0,1)$-projection $\tilde{u} \in \mathcal{P}^3(0,1)$ of u satisfies

$$\int_0^1 \tilde{u}(t) v(t)\, dt = \int_0^1 u(t) v(t)\, dt \qquad \forall v \in \mathcal{P}^3(0,1)$$

Ansatz $\tilde{u}(t) = \sum_{j=0}^{3} \tilde{\xi}_j t^j$. Let $v(t) = t^i$, $i = 0, \dots, 3$

$$\Rightarrow \sum_{j=0}^{3} \tilde{\xi}_j \underbrace{\int_0^1 t^j t^i \, dt}_{=: a_{ij}} = \underbrace{\int_0^1 e^t t^i \, dt}_{=: b_i}$$

$$a_{ij} = \dots = \frac{1}{i+j+1}$$

$$b_0 = \int_0^1 e^t = e-1$$
$$b_1 = \dots = 1$$
$$b_2 = \dots = e-2$$
$$b_3 = \dots = -2e+6$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_0 \\ \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{\xi}_3 \end{bmatrix} = \begin{bmatrix} e-1 \\ 1 \\ e-2 \\ -2e+6 \end{bmatrix}$$

$$\Rightarrow \underline{\tilde{u}(t)} \simeq 0{,}991 + 1{,}0183t + 0{,}4212t^2 + 0{,}2786t^3$$

Galerkin appr.:

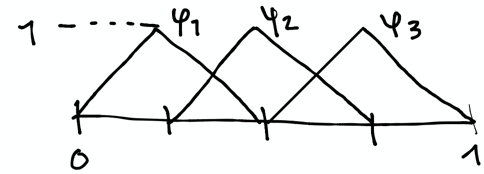
$$\underline{u_3(t)} \approx 1 + 1{,}0345t + 0{,}3879t^2 + 0{,}3017t^3$$

4) Consider the BVP

$$\begin{cases} -(a(x)\,u'(x))' = f(x), & 0<x<1 \\ u(0) = u(1) = 0 \end{cases}$$

for $a \equiv 1$ and $f = x$. Let $h = \frac{1}{4}$ and compute the Galerkin approximation cG(1) for u.

Solution



$$\varphi_1(x) = \begin{cases} 4x, & 0 \le x \le \frac{1}{4} \\ 4(\frac{1}{2} - x), & \frac{1}{4} \le x \le \frac{1}{2} \end{cases}$$

$$\varphi_3(x) = \begin{cases} 4(x - \frac{1}{2}), & \frac{1}{2} \le x \le \frac{3}{4} \\ 4(1-x), & \frac{3}{4} \le x \le 1 \end{cases}$$

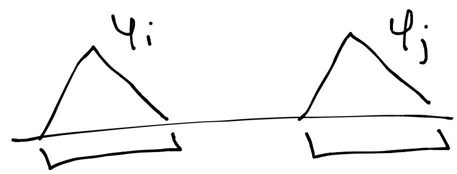
$$\varphi_2(x) = \begin{cases} 4(x - \frac{1}{4}), & \frac{1}{4} \le x \le \frac{1}{2} \\ 4(\frac{3}{4} - x), & \frac{1}{2} \le x \le \frac{3}{4} \end{cases}$$

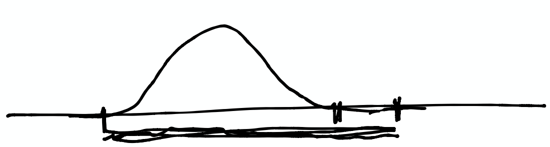
DE: $-u'' = x$ Multiply by $v \in \{\varphi_1, \varphi_2, \varphi_3\}$ and integrate over $x \in [0,1]$.

$-\int u''v = \int xv$ $\{P.I.\} \Rightarrow$ $\int u'v' = \int xv$

Let $u_h = \sum_{i=1}^{3} \xi_i \varphi_i(x)$

$$\sum_{i=1}^{3} \xi_i \underbrace{\int_0^1 \varphi_i' \varphi_j'}_{a_{ji}} = \underbrace{\int_0^1 x \varphi_j}_{b_j}, \qquad j = 1,2,3$$





$$a_{jj} = \int_{x_{j-1}}^{x_{j+1}} (\varphi_j'(x))^2 \, dx = \frac{2}{h} = 8$$

$$a_{j,j+1} = \dots = -\frac{1}{h} = -4 \overset{\text{symmetry}}{=} a_{j+1,j}$$

$a_{i,j}$ where $|i-j| \geq 2$ are 0 since they lack common support.

$$\Rightarrow A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

$$b_j = \int_0^1 \varphi_j(x) \cdot x \, dx = \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h} x \, dx + \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{h} x \, dx =$$

$$= \dots = h^2 j = \frac{j}{16} \quad \Rightarrow b = \left[\frac{1}{16} \;\; \frac{1}{8} \;\; \frac{3}{16}\right]^T$$

Solving $A\xi = b$ gives us $\xi = \left[\frac{5}{128} \;\; \frac{1}{16} \;\; \frac{7}{128}\right]^T$

$\Rightarrow$ Galerkin/cG(1) approximation is

$$u_h(x) = \frac{5}{128}\varphi_1(x) + \frac{1}{16}\varphi_2(x) + \frac{7}{128}\varphi_3(x)$$

5) i) Prove that $V^{(q)} := \mathrm{span}\{\sin(i\pi x),\ 1 \leq i \leq q\}$ is a subspace of $C_0([0,1]) := \{f \in C([0,1]) : f(0) = f(1) = 0\}$

ii) Show that $\{\sin(i\pi x)\}_{i=1}^{q}$ forms an orthogonal basis of $V^{(q)}$.

Solution

i) Obviously $V^{(q)} \subset C_0([0,1])$

$f, g \in V^{(q)} \Rightarrow \alpha f + \beta g \in V^{(q)} \quad \forall \alpha, \beta \in \mathbb{R}$.

ii) For basis we need linear independence, but this follows from orthogonality. Suffices to show orthog.

$$\int_0^1 \sin(i\pi x)\sin(j\pi x)\,dx =$$

$$= \left[-\frac{\cos(i\pi x)}{i\pi}\sin(j\pi x)\right]_0^1 + \int_0^1 \frac{\cos(i\pi x)}{i\pi}\cos(j\pi x)\cdot j\pi\,dx =$$

$$= \frac{j}{i}\int_0^1 \cos(i\pi x)\cos(j\pi x)\,dx =$$

$$= \frac{j}{i}\left[\frac{\sin(i\pi x)}{i\pi}\cos(j\pi x)\right]_0^1 + \frac{j}{i}\int_0^1 \frac{\sin(i\pi x)}{i\pi}\sin(j\pi x)\cdot j\pi\,dx =$$

$$= \frac{j^2}{i^2}\int_0^1 \sin(i\pi x)\sin(j\pi x)\,dx$$

$$\underbrace{\left(1 - \frac{j^2}{i^2}\right)}_{j \neq i \Rightarrow\ \neq 0} \int_0^1 \sin(i\pi x)\sin(j\pi x)\,dx = 0$$

$$\Rightarrow \int_0^1 \sin(i\pi x)\sin(j\pi x)\,dx = 0$$

Thus, orthogonal and linearly independent. $\checkmark$

2.5) Consider the problem:

Find an appr. sol. $U(x)$ to

$$\begin{cases} -u''(x) = 1 & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

using the Ansatz $U(x) = A\sin(\pi x) + B\sin(2\pi x)$.

a) calculate the exact sol.

b) Write down the residual $R(x) = -U''(x) - 1$

c) Use the orthogonality condition

$$\int_0^1 R(x)\sin(\pi n x)\,dx = 0 \quad , \quad n = 1, 2$$

to determine A and B.

d) plot the error $e(x) = u(x) - U(x)$.

Solution

a) $u' = -x + a$

$u = \frac{-x^2}{2} + ax + b$

$u(0) = 0 \Rightarrow b = 0 \Rightarrow u = \frac{-x^2}{2} + ax$

$u(1) = 0 \Rightarrow \frac{1}{2} = a \Rightarrow u = \frac{-x^2}{2} + \frac{x}{2}$

b) $U(x) = A\sin(\pi x) + B\sin(2\pi x)$

$U'(x) = A\pi\cos(\pi x) + 2B\pi\cos(2\pi x)$

$U''(x) = -A\pi^2\sin(\pi x) - 4B\pi^2\sin(2\pi x)$

$R(x) = A\pi^2\sin(\pi x) + 4B\pi^2\sin(2\pi x) - 1$

c) $$\int_0^1 \left(A\pi^2\sin(\pi x)+4B\pi^2\sin(2\pi x)-1\right)\sin(n\pi x)\,dx=0 \quad (I)$$

$n=1,2.$

<u>$n=1$</u>

$$(I) = \{\text{ortog.}\} = \int_0^1 \left(A\pi^2\sin^2(\pi x)-\sin(\pi x)\right)dx = 0$$

$$A\pi^2 \underbrace{\int_0^1 \sin^2(\pi x)\,dx}_{=\frac{1}{2}} = \underbrace{\int_0^1 \sin(\pi x)\,dx}_{=\frac{2}{\pi}} \Rightarrow \frac{A\pi^2}{2} = \frac{2}{\pi}$$

$$A = \frac{4}{\pi^3}$$

<u>$n=2$</u>

$$(I) = \{\text{ortog.}\} = \int_0^1 \left(4B\pi^2\sin^2(2\pi x)-\sin(2\pi x)\right)dx = 0$$

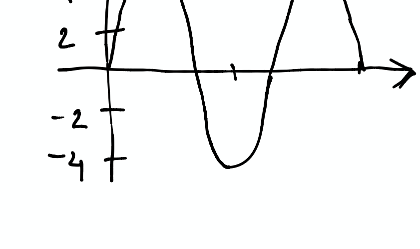
$$4B\pi^2 \int_0^1 \sin^2(2\pi x) = \int_0^1 \sin(2\pi x)$$

$$\ldots \Rightarrow \frac{4B\pi^2}{2} = 0 \Rightarrow B=0$$

$$U(x) = \frac{4}{\pi^3}\sin(\pi x)$$

This is the same as the L^2-projection of $u(x)$ onto $\mathrm{span}\{\sin \pi x, \sin 2\pi x\}$

d) Plot of error:



2.7) Let $U = \xi_0 \phi_0(x) + \xi_1 \phi_1(x)$ be an appr. sol. to

$$\begin{cases} -u''(x) = x-1 & 0<x<\pi \\ u'(0) = u(\pi) = 0 \end{cases}$$

$$\phi_0(x) = \cos\left(\frac{x}{2}\right) \quad \phi_1 = \cos\left(\frac{3x}{2}\right)$$

a) Find analytical sol.

b) Define residual $R(x)$

c) Compute ξ_i using the orthog. cond.:

$$\int_0^\pi R(x)\phi_i(x)\,dx = 0 \quad , \quad i=0,1$$

Solution

a) $u' = x - \frac{x^2}{2} + a$

$$u = \frac{x^2}{2} - \frac{x^3}{6} + ax + b$$

$$u'(0) = 0 \Rightarrow a = 0$$

$$u(\pi) = 0 \Rightarrow b = \frac{\pi^3}{6} - \frac{\pi^2}{2} = \frac{\pi^3 - 3\pi^2}{6}$$

$$\underline{u(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{\pi^3 - 3\pi^2}{6}}$$

b) $U'' = \frac{-\xi_0}{4}\cos\frac{x}{2} - \frac{9\xi_1}{4}\cos\frac{3x}{2}$

$$\underline{R(x) = U'' + x - 1}$$

This is the same as the L^2 projection onto span$\{\cos\frac{x}{2}, \cos\frac{3x}{2}\}$

c) $\int_0^\pi \left(-\frac{\xi_0}{4} \cos\frac{x}{2} - \frac{9\xi_1}{4}\cos\frac{3x}{2} + x - 1 \right) \cos\frac{nx}{2}\, dx = 0 \quad n = 1, 3$

<u>$n=1$</u> $\int_0^\pi \left(-\frac{\xi_0}{4} \cos^2\left(\frac{x}{2}\right) + (x-1)\cos\left(\frac{x}{2}\right)\right) dx = 0$

$\ldots \Rightarrow -\xi_0 \frac{\pi}{8} + 2(\pi - 3) = 0 \quad \Rightarrow \xi_0 = \frac{16(\pi-3)}{\pi}$

<u>$n=3$</u> $\int_0^\pi \left(\frac{-9\xi_1}{4} \cos^2\left(\frac{3x}{2}\right) + (x-1)\cos\left(\frac{3x}{2}\right)\right) dx = 0$

$\ldots \Rightarrow -\frac{9\pi}{8}\xi_1 + \frac{2}{9} - \frac{2\pi}{3} = 0 \quad \Rightarrow \xi_1 = \frac{8(2-6\pi)}{81\pi}$

correction

7) Let $V^{(q)} = P^{(q)}(0,1)$ polyn. of deg $\leq q$.

Define $V_0^{(q)} = \{ v \in V^{(q)} : v(0) = 0 \}$

Prove that $V_0^{(q)}$ is a subsp. of $V^{(q)}$.

<u>Solution</u>

- $0 \in V_0^{(q)}$
- $(u+v)(0) = u(0) + v(0) = 0$
- $(\alpha u)(0) = \alpha u(0) = 0$

$\checkmark$