B

• FEM:

BVP

↓

VF Find $u \in V$ s.t. $a(u,v) = \ell(v)$ $\forall v \in \hat{V}$

↓

Galerkin

cG(1)

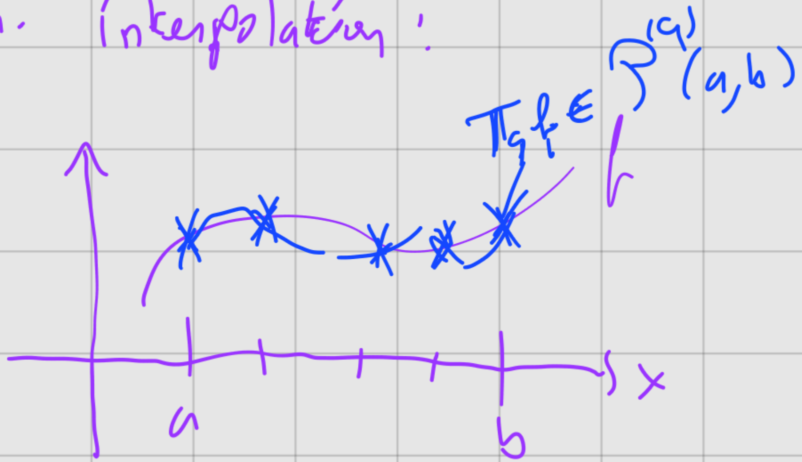
FE problem

Find $u_h \in V_h$ s.t. $a(u_h, v_h) = \ell(v_h)$ $\forall v_h \in \hat{V}_h$

↓

linear syst. $Ax = b$

• Polyn. interpolation:



• $\underline{u_t(x,t)} - \underline{u_{xx}(x,t)} = \underline{f(x,t)}$

num. methods — IVP

BVP → FEM

• Notation, quiz, cpu lab

Ex: Linear interpolation on $[0,1]$: classical basis

Here, $q=1 \Rightarrow x_0=0$ and $x_1=1$.

Using $\Pi_1 f \in \mathcal{P}^{(1)}(0,1) = \text{span}(1,x)$, one can write

$$\Pi_1 f(x) = a_0 \cdot 1 + a_1 \cdot x$$

$\leftarrow$ basis

unknown coeff.

Since we look for an interpolant, we have the conditions:

$$\underbrace{f(x_0)}_{\text{given}} \overset{!}{=} \Pi_1 f(x_0) = \Pi_1 f(0) = a_0 \quad \rightarrow a_0 = f(x_0)$$

$$f(x_1) \overset{!}{=} \Pi_1 f(x_1) \underset{\text{Def } x_0, x_1}{=} \Pi_1 f(1) \underset{\text{Def } \Pi_1 f}{=} a_0 + a_1$$

$$\rightarrow a_1 = f(x_1) - f(x_0)$$

$$\Rightarrow \Pi_1 f(x) = f(x_0) \cdot 1 + (f(x_1) - f(x_0)) \cdot x \quad // \quad :-)$$

(eq. of line)

One can do the same with another basis:

Def: Lagrange polynomials are given by

$$\lambda_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{q} \frac{x - x_j}{x_i - x_j} \qquad i = 0, 1, 2, \dots, q$$

Rem:
- $\lambda_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$
- $\mathcal{P}^{(q)}(a,b) = \operatorname{span}(\lambda_0(x), \lambda_1(x), \dots, \lambda_q(x))$

Ex: Linear interpolant on $[0,1]$: Lagrange basis / nodal basis

linear $\to q = 1 \to x_0 = 0$ and $x_1 = 1$.

Now, we write $\Pi_1 f(x) = \underbrace{b_0}_{} \overbrace{\lambda_0(x)}^{\text{basis}} + \underbrace{b_1}_{\text{unknown coeff.}} \cdot \overbrace{\lambda_1(x)}^{}$

with Lagrange polyn.

$$\lambda_0(x) \overset{\text{Def } \lambda_j}{=} \frac{x - x_1}{x_0 - x_1} \overset{\text{Def } x_0, x_1}{=} \frac{x-1}{0-1} = 1 - x$$

$$\lambda_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{1 - 0} = x$$

Again, we have the 2 conditions $f(x_0) = \Pi_1 f(x_0)$ and $f(x_1) = \Pi_1 f(x_1)$ and get, as before,

$$\Pi_1 f(x) = f(x_0) + x\,(f(x_1) - f(x_0)) \quad //$$

(same as before)

Ex: $q=2$ → book p. 92

What is the error of the linear interpolant?

Recall: Norms $\|f\|_{L^p(a,b)} = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$ $\quad p = 1, 2$

$$\|f\|_{L^\infty(a,b)} = \sup_{x \in [a,b]} |f(x)|$$

Gives way to measure "distance" between functions: $\|f - g\|_{L^p(a,b)}$

Th: Let $p = 1, 2, \infty$. Assume $f \in \mathcal{C}^2(a,b)$.
Then, there exists constants C_1, C_2, C_3 s.t.

1) $\|\Pi_1 f - f\|_{L^p(a,b)} \leq C_1 \cdot (b-a)^{\boxed{2}} \cdot \|f''\|_{L^p(a,b)}$

2) $\| \Pi_1 f - f \|_{L^p(a,b)} \leq C_2 \cdot (b-a) \cdot \| f' \|_{L^p(a,b)}$

3) $\| (\Pi_1 f)' - f' \|_{L^p(a,b)} \leq C_3 \cdot (b-a) \cdot \| f'' \|_{L^p(a,b)}$

for $p = 1, 2, \infty$.

Proof:

- From the previous expl. we know that $\Pi_1 f(x) = f(x_0) \cdot \lambda_0(x) + f(x_1) \cdot \lambda_1(x)$, where

$$x_0 = a, \ x_1 = b, \ \lambda_0(x) = \frac{x-b}{a-b}, \ \lambda_1(x) = \frac{x-a}{b-a}.$$

- Do a Taylor expansions of $f(x_0)$ and $f(x_1)$ around x:

$$f(x_0) = f(x) + (x - x_0) f'(x) + \frac{(x-x_0)^2}{2!} f''(\xi_0), \text{ for some } \xi_0 \in (a, x)$$

$$f(x_1) = f(x) + (x - x_1) f'(x) + \frac{(x-x_1)^2}{2!} f''(\xi_1), \text{ for some } \xi_1 \in (x, b)$$

- Inserting everything into $\Pi_1 f(x)$ gives

$$\Pi_1 f(x) = f(x_0) \lambda_0(x) + f(x_1) \cdot \lambda_1(x) = \ldots =$$

$$= f(x) + \frac{1}{2}(x_0 - x)^2 f''(\xi_0)\,\lambda_0(x) + \frac{1}{2}(x_1 - x)^2 f''(\xi_1)\cdot\lambda_1(x)$$

Hence, the error reads

$$|\Pi_1 f(x) - f(x)| \le \frac{1}{2} \underbrace{|x_0 - x|^2}_{\le (b-a)^2} \overbrace{|f''(\xi_0)|}^{\le \max_{x\in[a,b]} |f''(x)|} \underbrace{|\lambda_0(x)|}_{\le 1} +$$

$$+ \frac{1}{2} \underbrace{|x_1 - x|^2}_{\le (b-a)^2} \underbrace{|f''(\xi_1)|}_{\le \max_{x\in[a,b]} |f''(x)|} \cdot \underbrace{|\lambda_1(x)|}_{\le 1}$$

since $x \in (a,b)$

$$\le 1 \cdot (b-a)^2 \cdot \underbrace{\max_{y\in[a,b]} |f''(y)|}_{\|f''\|_{L^\infty(a,b)}} \quad (\text{Def})$$

The above bound is valid for all $x \in [a,b]$, hence one has:

$$\|\Pi_1 f - f\|_{L^\infty(a,b)} \le (b-a)^2 \cdot \|f''\|_{L^\infty(a,b)}$$

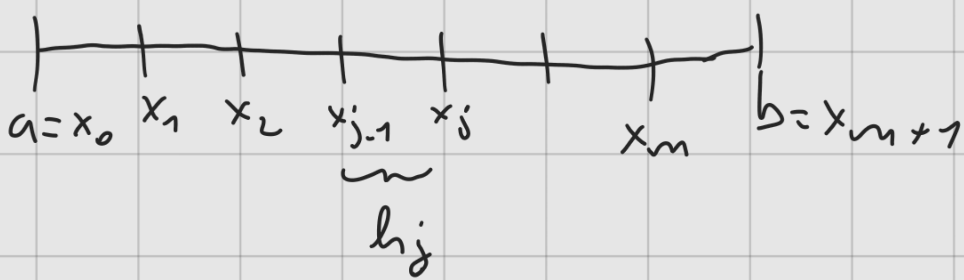
$(p = \infty, 1))$ ∎

c) Continuous piecewise linear interpolation:

Here, we consider cont. pw linear int. instead of linear int.

Recall: $V_h(a,b) = \mathrm{span}(\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_{m+1})$ with hat fct φ_j.

T_h:



Each $v \in V_h(a,b)$ can be written

$$v(x) = \sum_{j=0}^{m+1} \underbrace{\xi_j}_{= v(x_j)} \varphi_j(x)$$

Def: The mesh function is given by

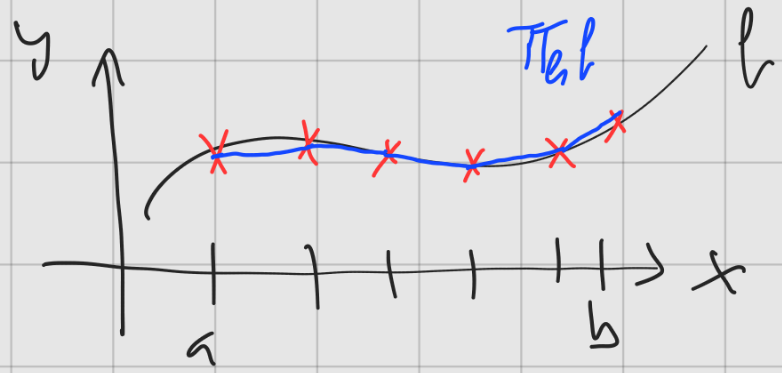
$h(x) = h_j$ for $x \in (x_{j-1}, x_j)$

(usefull for non-uniform partition)

Def: Let $f: [a,b] \to \mathbb{R}$ contienous and the above partition T_h. The continuous piecewise linear interpolant of f reads

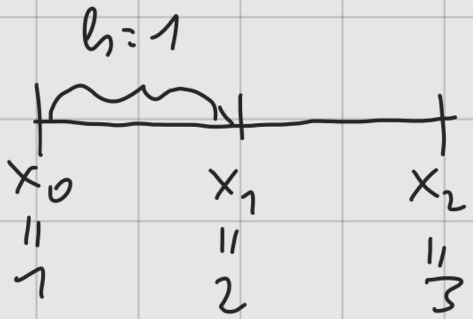
$$\Pi_h f(x) = \sum_{j=0}^{m+1} f(x_j) \cdot \varphi_j(x) \in V_h(a,b)$$

Rem: Important def since FE sol. is someone living in $V_h(a,b)$, see error analysis of FEM (later).



Ex: Let $f(x) = (x-1)\cdot(x-2)+1$ for $x \in [1,3]$ and uniform partition of $[1,3]$ into 2 subintervals. What is $\Pi_h f$?

Partition:



$m = 1$

$$h = \frac{b-a}{m+1} = \frac{3-1}{1+1} = 1$$

The interpolant reads

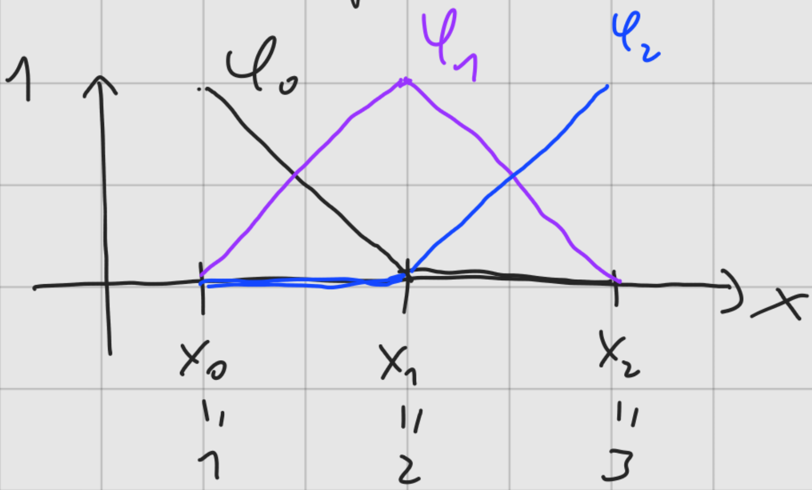
$$\Pi_h f(x) = f(x_0)\cdot\varphi_0(x) + f(x_1)\cdot\varphi_1(x) + f(x_2)\cdot\varphi_2(x),$$

where $f(x_0) = f(1) = 1$ (using Def x_0, f)

$f(x_1) = f(2) = 1$

$f(x_2) = f(3) = 3$

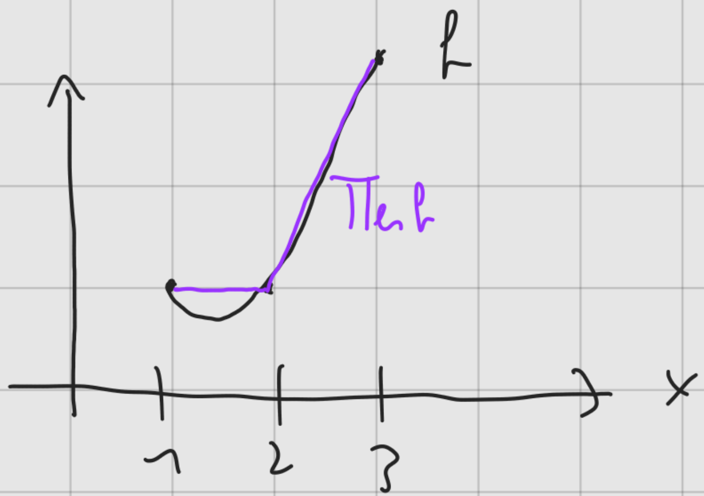
and the hat fct.



Hence,

$$\Pi_h f(x) \overset{\text{Def } f/\text{above}}{=} \varphi_0(x) + \varphi_1(x) + 3\cdot\varphi_2(x) \overset{\text{Def } \varphi_j}{=}$$

$$= \begin{cases} \frac{x-x_1}{-h} + \frac{x-x_0}{h} + 0 = 1 & \text{for } x_0 \le x \le x_1 \\ 0 + \frac{x-x_2}{-h} + 3\left(\frac{x-x_1}{h}\right) = 2x-3 & \text{for } x_1 \le x \le x_2 \end{cases}$$



$h \to 0$
$m \to \infty$ $\Rightarrow$ better interpolation

What is the error of this interpolant?

Th: Let $f \in C^2(a,b)$ and $\Pi_h f$ the above pw linear interpolant on uniform partition. Then, $\exists$ constants C_1, C_2, C_3 s.t.

1) $\| \Pi_h f - f \|_{L^p(a,b)} \leq C_1 \cdot h^{\boxed{2}} \cdot \| f'' \|_{L^p(a,b)}$

2) $\| \Pi_h f - f \|_{L^p(a,b)} \leq C_2 \cdot h \cdot \| f' \|_{L^p(a,b)}$

3) $\| (\Pi_h f)' - f' \|_{L^p(a,b)} \leq C_3 \cdot h \cdot \| f'' \|_{L^p(a,b)}$

for $p = 1, 2, \infty$.

Rem:
- If $h \to 0$, then error $\to 0$
- f'' LARGE $\Rightarrow$ f bends a lot, linear interpolant not so good.
- For non-uniform partition, one uses the mesh function and gets

$$\| \Pi_h f - f \|_{L^p(a,b)} \leq C_1 \cdot \| h^2 f'' \|_{L^p(a,b)}$$

f. instance

Proof of 1) and p = 1,2)

$$\|\Pi_h f - f\|^p_{L^p(a,b)} \overset{\text{Def norm}}{=} \int_a^b |\Pi_h f(x) - f(x)|^p dx \overset{\text{Def partition}}{=}$$

$$= \sum_{j=1}^{m+1} \int_{x_{j-1}}^{x_j} |\Pi_h f(x) - f(x)|^p dx = \sum_{j=1}^{m+1} \|\underbrace{\Pi_h f}_{\text{polyn. of degree 1 on } (x_{j-1}, x_j)} - f\|^p_{L^p(x_{j-1}, x_j)}$$

Def norm

Previous Th.

$$\leq (C \cdot h^2)^p \sum_{j=1}^{m+1} \|f''\|^p_{L^p(x_{j-1}, x_j)}$$

$$\leq (C \cdot h^2)^p \|f''\|^p_{L^p(a,b)} .$$

Taking power $\frac{1}{p}$ gives us the first point of the theorem. ∎