R

- Polyn. interpolant $\Pi_q f \in \mathcal{P}^{(q)}(a,b) = \text{span}(\lambda_j)$
  Lagrange
- PW linear interpolant $\Pi_h f \in V_h := \text{span}(\varphi_0, \varphi_1, \dots, \varphi_{m+1})$
  hat

Errors

$$\|\Pi_h f - f\|_{L^p} \leq C \cdot h^2 \|f''\|_{L^p}$$ uniform partition

$$\|(\Pi_h f)' - f'\|_{L^p} \leq C \cdot h \cdot \|f''\|_{L^p}$$ — " —

- FEM for Poisson $\longrightarrow$ linear syst. of eq.
  $S \cdot \xi = b$, where
  $b = (b_j)_{j=1}^{m}$ with $\underbrace{b_j = \int_0^1 f(x) \cdot \varphi_j(x)\,dx}_{\text{difficult to compute exactly.}}$

3) Numerical integration / quadrature rules:

Goal: Find numerical approximation of

$$\int_a^b f(x)\,dx$$

Idea: $f(x) \approx \Pi_q f(x)$ with polyn. interpolant

$$\underbrace{\int_a^b f(x)\,dx}_{\text{difficult to integrate exactly.}} \approx \underbrace{\int_a^b \Pi_q f(x)\,dx}_{\text{"easy" to integrate exactly}}$$

The following are classical exples of quadrature rules / formulas:

(i) $q=0$: $f(x) \approx f\left(\frac{a+b}{2}\right)$ constant polyn.

Hence, $\int_a^b f(x)\,dx \approx \int_a^b \underbrace{f\left(\frac{a+b}{2}\right)}_{\text{constant.}}\,dx = (b-a)\cdot f\left(\frac{a+b}{2}\right)$.

This is the midpoint rule $\int_a^b f(x)\,dx \approx (b-a)\cdot f\left(\frac{a+b}{2}\right)$

(ii) $q=1$: $f(x) \approx f(a)\cdot\lambda_0(x) + f(b)\cdot\lambda_1(x)$, where

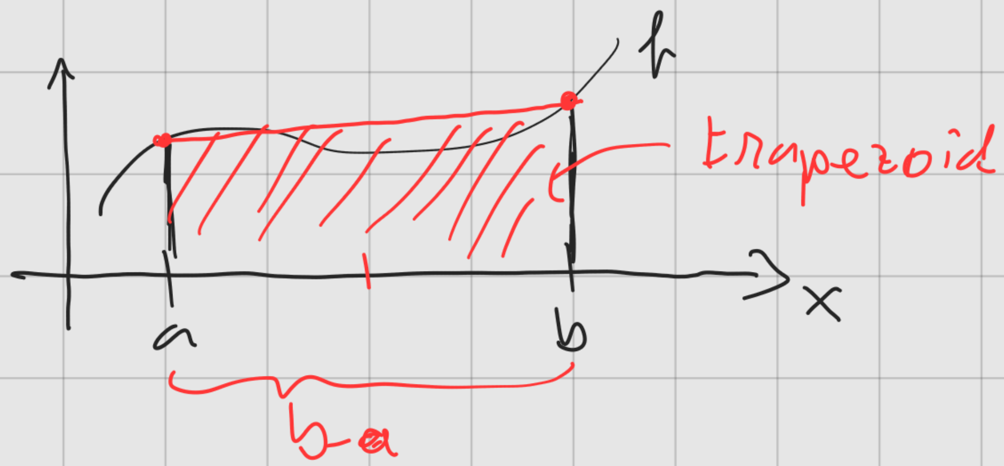
$\lambda_0(x) = \frac{x-b}{a-b}$ and $\lambda_1(x) = \frac{x-a}{b-a}$ Lagrange polyn.

Hence, $\int_a^b f(x)dx \approx f(a) \underbrace{\int_a^b \lambda_0(x)\,dx}_{\text{exact}} + f(b) \cdot \underbrace{\int_a^b \lambda_1(x)\,dx}_{\text{exact}} =$

$$= \dots = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

This is the trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left(f(a)+f(b)\right)$$



(iii) $q=2$: $f(x) \approx \Pi_2 f(x)$ ... details book ...
p. 97

Similarly, one gets Simpson's rule

$$\int_a^b f(x)dx \approx \frac{b-a}{6} \cdot \left(f(a) + 4 \cdot f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Ex: What are the approx. of $\int^{\pi/4} \sin(x)dx$

given by above quadrature rules?

Midpoint $\displaystyle\int_0^{\pi/4} \sin(x)\,dx \approx \left(\frac{\pi}{4}-0\right)\cdot \sin\left(\frac{\pi}{8}\right) \approx 0.3$

Trapezoidal $\displaystyle\int_0^{\pi/4} \sin(x)\,dx \approx \frac{(\pi/4-0)}{2}\left(\sin\left(\frac{\pi}{4}\right)+\sin(0)\right)$
$\approx 0{,}27$

Simpson : ≈ 0.2929

Exact integral : $\displaystyle\int_0^{\pi/4} \sin(x)\,dx = 0{,}29289\ldots$

⚠ In practice : $\displaystyle\int_a^b f(x)\,dx$

One first consider a partition

$a = x_0 < x_1 < x_2 \ldots \leq x_N = b$, where

$x_j - x_{j-1} = h_j$ small (N LARGE)

Next, $\displaystyle\int_a^b f(x)\,dx = \sum_{j=0}^{N-1} \underbrace{\int_{x_j}^{x_{j+1}} f(x)\,dx}$

apply quadrature rule (f.ex. midpoint) on these "small" integrals

This procedure is called

composite quadrature rule

Chapter V: FEM for BVP

Goal: Present and analyse FEM for BVPs.

1) Model problem:

Prob: Let $a, f : [0,1] \to \mathbb{R}$ with $a(x) \geq \alpha_0 > 0$ and a, f are continuous ($f \in L^2$, a pw cont./bnd)
Consider model problem

$$(BVP) \quad \begin{cases} -(a(x)u'(x))' = f(x) & \text{for } 0 < x < 1 \\ u(0) = 0 = u(1) \end{cases}$$

We obtain the variational formulation

$$(VF) \quad \text{Find } u \in H_0^1 \text{ s.t. } \int_0^1 a(x) u'(x) v'(x) dx = \int_0^1 f(x) v(x) dx \quad \forall v \in H_0^1,$$

where $H_0^1 = \{ v \in H^1 \text{ s.t. } u(0) = u(1) = 0 \}$

Next, we get the FE problem

(FE) Find $u_h \in V_h^0$ s.t. $\int_0^1 a(x) u_h'(x) v_h'(x)\,dx = \int_0^1 f(x) v_h(x)\,dx \quad \forall v_h \in V_h^0$,

where $V_h^0 = \text{span}(\varphi_1, \varphi_2, \dots, \varphi_m)$ and

$$T_h : 0 = x_0 < x_1 < \dots < x_{m+1} = 1, \quad x_j - x_{j-1} = h_j$$

Mesh function $h(x) = h_j$ for $x \in [x_{j-1}, x_j)$

↳ pw constant

⚠ Obs. that $V_h^0 \subset H_0^1$ and then consider $v_h \in V_h^0 \subset H_0^1$ as a test fct in (VF) :

$$\text{"}(VF) \overset{\text{substraction}}{-} (FE)\text{"} : \int_0^1 a(x)\left(u'(x) - u_h'(x)\right) v_h'(x)\,dx = 0 \quad \forall v_h \in V_h^0 \quad (GO)$$

This relation is called Galerkin orthogonality :

the error of FEM $u - u_h$ is orthogonal to V_h^0 in the energy inner product, that we define next.

[Simpson p.97]

Def: For $f, g \in H_0^1$ and $a(\cdot)$ as above, we define

- the weighted inner product: $(f,g)_a := \int_0^1 a(x) \cdot f(x) \cdot g(x) \, dx$

  L^2_a [$a \equiv 1 \rightsquigarrow L^2$ space]

- the energy inner product: $(f,g)_E = (f', g')_a$

- with the corresponding norms:

  $\|f\|_a = \sqrt{(f,f)_a}$ and $\|f\|_E = \sqrt{(f,f)_E} = \sqrt{(f',f')_a}$

Rem: The above definition must be adapted for other problems.

2) A priori error estimates in energy norm:

Rem: A priori $\longleftrightarrow$ error FEM $\leq C(h)$, where the fct $C(\cdot)$ depends on exact sol. u but not on u_h

We first show that cG(1) (FE approx. u_h) is the best approx. of u in V_h^0 in the energy norm.

Thm: Let u be the sol. to (VF) and u_h sol. to (FE)

Th: Let u be the sol. to (VF) and u_h sol. to (FE)

One has:

$$\|u-u_h\|_E \leq \|u-v\|_E \quad \forall v \in V_h^0$$

Proof:

$$\|u-u_h\|_E^2 \underset{\text{Def energy norm}}{=} (u-u_h, u-u_h)_E \overset{v \in V_h^0}{=} (u-u_h, u-v+v-u_h)_E =$$

$$\overset{\text{linearity}}{=} (u-u_h, u-v)_E + \underbrace{(u-u_h, v-u_h)_E}_{\substack{=0 \text{ by (GO) since} \\ u-u_h \perp v-u_h \in V_h^0}} \overset{\text{C-S}}{\leq} \|u-u_h\|_E \cdot \|u-v\|_E$$

Hence, $\|u-u_h\|_E \leq \|u-v\|_E \quad \forall v \in V_h^0$:-) ∎

Th. (A priori error estimates)

Let u, u_h be sol. to (VF), resp. (FE).

Assume that $u'' \in L_a^2(0,1) = \{v : [0,1] \to \mathbb{R} : \|v\|_a < \infty\}$

Then, $\exists$ $C > 0$ s.t.

$$\|u-u_h\|_E \leq C \cdot \|h \cdot u''\|_a$$

Rem: • For uniform partition $h(x) = h$ constant

$$\rightsquigarrow \|u-u_h\|_E \leq C \cdot \boxed{h} \, \|u''\|_a \xrightarrow[h \to 0]{} 0$$

- Further analysis gives: $\|u-u_h\|_{L^2} \leq C \cdot h^2 \|u\|_{H^2}$

<u>Proof</u>: Let us start with

$$\|u-u_h\|_E \underset{\substack{\uparrow \\ \text{previous Th.} \\ \text{FE = best approx.}}}{\leq} \|u - \underbrace{\Pi_h u}_{\substack{\text{cont. pw linear} \\ \text{interpolant of } u}}\|_E \overset{\substack{\text{Def norm} \\ \downarrow}}{=} \left(\int_0^1 a(x) \left(u'(x) - (\Pi_h u)'(x)\right)^2 dx \right)^{1/2}$$

$$\leq \left(\max_{x \in [0,1]} |a(x)| \right)^{1/2} \cdot \underbrace{\left(\int_0^1 \left(u'(x) - (\Pi_h u)'(x)\right)^2 dx \right)^{1/2}}_{\underbrace{\|u' - (\Pi_h u)'\|_{L^2(0,1)}}_{\leq C \cdot \|h u''\|_{L^2(0,1)}}}$$

(error of interpolant, previous Chapter)

get "$a(\cdot)$" back

$$\leq C \cdot \left(\max_x |a(x)| \right)^{1/2} \cdot \left(\int_0^1 \frac{a(x)}{\min_{x \in [0,1]} |a(x)|} h(x)^2 \left(u''(x)\right)^2 dx \right)^{1/2}$$

$\min_{x \in [0,1]} |a(x)| \leq a(x)$

$$\leq C \cdot \left(\frac{\max_{[0,1]} |a(x)|}{\min_{[0,1]} |a(x)|} \right)^{1/2} \cdot \left(\int_0^1 a(x) h^2(x) \left(u''(x)\right)^2 dx \right)^{1/2}$$

$$\underbrace{}_{\hat{C}} \qquad \underbrace{}_{\|h u''\|_a} \quad \text{by Def norm}$$

$\Rightarrow \|u-u_h\|_E \leq \hat{C} \cdot \|h u''\|_a$:-)

[B]

3) A posteriori error estimates in energy norm!

Rem: A posteriori $\to$ error depends on u_h but not on u

Th: Under technical assumptions on u, u_h the exact sol to (VF), resp. (FE).

Let

$$R(u_h(x)) = f(x) + (a(x)u_h'(x))'$$

be residual of the problem. One then has the a posteriori error estimates

$$\|u-u_h\|_E \leq C \cdot \left(\int_0^1 \frac{1}{a(x)} \cdot h^2(x) \cdot R^2(u_h(x))\, dx \right)^{1/2}$$