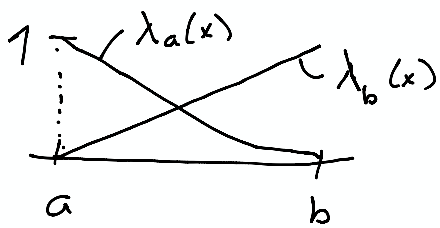
6) Prove the $L^p(a,b)$ error estimates for the interpolation with p=1 and p=2:

$$\|f - \Pi_1 f\|_{L^p(a,b)} \le (b-a)^2 \|f''\|_{L^p(a,b)}.$$

Solution

$$\Pi_1 f(x) = f(a)\lambda_a(x) + f(b)\lambda_b(x)$$



$$\lambda_a(x) = \frac{b-x}{b-a} \qquad \lambda_a'(x) = \frac{-1}{b-a}$$

$$\lambda_b(x) = \frac{x-a}{b-a} \qquad \lambda_b'(x) = \frac{1}{b-a}$$

useful identities: $\lambda_a + \lambda_b = 1$

$$a\lambda_a + b\lambda_b = x$$

$$(a-x)\lambda_a + (b-x)\lambda_b = 0$$

Assume $f \in C^k([a,b])$ and $f^{(k)}$ abs. cont. on $[a,b]$.
The Taylor expansion around a point c is

$$f(x) = f(c) + f'(c)(x-c) + \dots + \frac{f^{(k)}(c)(x-c)^k}{k!} + R_k(x),$$

$$R_k(x) = \int_c^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k \, dt.$$

Then for a point $x \in [a,b]$, we have:

$$f(a) = f(x) + f'(x)(a-x) + \int_x^a f''(t)(a-t)\,dt$$

$$f(b) = f(x) + f'(x)(b-x) + \int_x^b f''(t)(b-t)\,dt$$

$$\Rightarrow f(x) - \Pi_1 f(x) = f(x) - \lambda_a(x) f(a) - \lambda_b(x) f(b) =$$

$$= f(x) - \left[\lambda_a(x) f(x) + \lambda_a(x) f'(x)(a-x) + \lambda_a(x)\int_x^a f''(t)(a-t)\,dt\right] -$$

$$- \left[\lambda_b(x) f(x) + \lambda_b(x) f'(x)(b-x) + \lambda_b(x)\int_x^b f''(t)(b-t)\,dt\right]$$

$$= f(x)\underbrace{\left[1 - \lambda_a(x) - \lambda_b(x)\right]}_{=0} - f'(x)\underbrace{\left[\lambda_a(x)(a-x) + \lambda_b(x)(b-x)\right]}_{=0} -$$

$$-\lambda_a(x)\int_x^a f''(t)(a-t)\,dt - \lambda_b(x)\int_x^b f''(t)(b-t)\,dt =$$

$$= -\lambda_a(x)\int_a^x f''(t)(t-a)\,dt - \lambda_b(x)\int_x^b f''(t)(b-t)\,dt$$

$$\Downarrow$$

$$|f(x) - \Pi_1 f(x)| \leq \{\Delta\text{-ineq.}\}$$

$$\leq |\lambda_a(x)| \int_a^x |f''(t)|\underbrace{|t-a|}_{\leq |b-a|}\,dt + |\lambda_b(x)| \int_x^b |f''(t)|\underbrace{|b-t|}_{\leq |b-a|}\,dt \leq$$

$$\leq \underbrace{\left(|\lambda_a(x)| + |\lambda_b(x)|\right)}_{=1} |b-a| \int_a^b |f''(t)|\,dt =$$

$$= |b-a| \int_a^b |f''(t)|\,dt.$$

$\underline{p=1}$ $\quad \|f-\Pi_1 f\|_{L^1(a,b)} \doteq \int_a^b |f-\Pi_1 f| \le$

$$\le \int_a^b |b-a| \int_a^b |f''(t)|\, dt\, dx = (b-a)^2 \|f''\|_{L^1(a,b)} .$$

$\underline{p=2}$ $\quad \|f-\Pi_1 f\|^2_{L^2(a,b)} = \int_a^b |f-\Pi_1 f|^2\, dx \le$

$$\le \int_a^b |b-a|^2 \left(\int_a^b 1 \cdot |f''(t)|\, dt \right)^2 dx \quad \le \{C-S\}$$

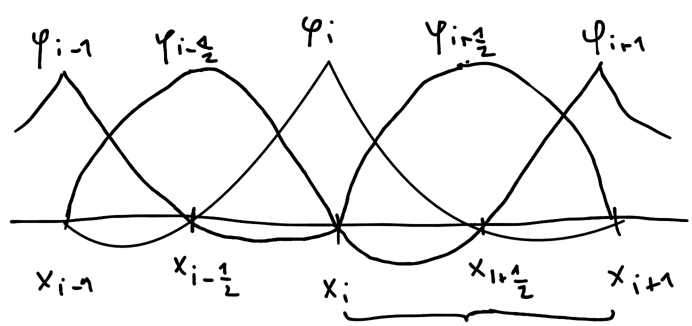
$$\le \int_a^b |b-a|^2\, dx \cdot \int_a^b 1^2\, dt \cdot \int_a^b |f''(t)|^2\, dt =$$

$$= (b-a)^4 \|f''\|^2_{L^2(a,b)} .$$

%.

7) Write down a basis for the set of p.w. quadratic polynomials $W_h^{(2)}$ on (a,b) and plot a sample of the function.

<u>Solution</u>



Lagrange polynomials

$I=(a,b)$ is partitioned equidistantly

$$a = x_0 < x_{\frac{1}{2}} < x_1 < \dots < x_{n-\frac{1}{2}} < x_n = b$$

with $x_i - x_{i-1} = h$ i.e. $x_i - x_{i-\frac{1}{2}} = \frac{h}{2}$

Our basis is then

$$\varphi_i(x) = \begin{cases} 2(x-x_{i-1})(x-x_{i-\frac{1}{2}})/h^2 & x \in [x_{i-1}, x_i] \\ 2(x_{i+\frac{1}{2}}-x)(x_{i+1}-x)/h^2 & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases}$$

$$\varphi_{i-\frac{1}{2}}(x) = \begin{cases} 4(x-x_{i-1})(x_i-x)/h^2 & x \in [x_{i-1}, x_i] \\ 0 & \text{else} \end{cases}$$

Use this basis for problem 2 in assignment 1.

Note: If you use a different grid, e.g. with $x_i - x_{i-\frac{1}{2}} = h$, the basis functions will differ from these! Take extra care that your basis functions evaluate to 1 (and 0) in the right points!

9) Let $I = (a,b)$ and consider a partition of I, $\{x_i\}_{i=0}^{m+1}$ with mesh function $h(x) = h_i = x_i - x_{i-1}$.

Prove that any value of f on the subinterval can be used to define $\pi_h f$ satisfying the error bound:

$$\|f - \pi_h f\|_{L^\infty(a,b)} \le \max_{1 \le i \le m+1} h_i \|f'\|_{L^\infty(I_i)} \le \|hf'\|_{L^\infty(a,b)}$$

where $h = \max_i h_i$. Prove that choosing the midpoint improves this bound by an extra factor of $\frac{1}{2}$.

<u>Solution</u> Let $I_i = (x_{i-1}, x_i]$, $1 \le i \le m+1$ and $\xi_i \in I_i$

Define $\Pi_h f(x) = \begin{cases} f(\xi_i), & x \in I_i \\ f(\xi_1), & x = x_0 \end{cases}$

For $x \in I_i$, we have

$$|f(x) - \Pi_h f(x)| = |f(x) - f(\xi_i)| = \left\{ \begin{array}{l} \text{Mean value thm:} \\ f'(\eta) = \dfrac{f(x) - f(\xi_i)}{x - \xi_i} \\ \text{Assume } f \in C, \text{ diff.able} \end{array} \right\}$$

$$= |f'(\eta)| \underbrace{|x - \xi_i|}_{(*)\ \le h_i} \le h_i \cdot \|f'\|_{L^\infty(I_i)} .$$

$$\therefore \|f - \Pi_h f\|_{L^\infty(I_i)} \le h_i \|f'\|_{L^\infty(I_i)}$$

$$\Rightarrow \|f - \Pi_h f\|_{L^\infty(a,b)} \le \max_{1 \le i \le m+1} \|f - \Pi_h f\|_{L^\infty(I_i)} \le$$

$$\le \max_{1 \le i \le m+1} h_i \|f'\|_{L^\infty(I_i)} \le \max_{1 \le i \le m+1} \|hf'\|_{L^\infty(I_i)} \le$$

$$\le \|hf'\|_{L^\infty(a,b)} .$$

If we choose ξ_i as the midpoint, then $(*) \le \frac{h_i}{2}$ instead and the second statement follows. ⁒

5.7) Consider the two-point BVP

$$(**)\quad \begin{cases} -u'' = 0 & 0 < x < 1 \\ u'(0) = 5 & u(1) = 0 \end{cases}$$

Let $\sigma_h = x_j = jh$, $j = 0, \ldots, N$, $h = \frac{1}{N}$ be a partition of the interval $0<x<1$ into N subintervals and let V_h be the corr. space of cont. p.w. linear fcns.

a) Use V_h, $N=3$ and formulate a FEM for $(**)$

b) Compute a FE appr. $U \in V_h$ if possible.

Solution

$V = \{ v \in H^1,\ v(1) = 0 \}$

VF: Find $u \in V$ s.t

$$-\int_I u'' v = 0 \qquad -\left[u'(x)v(x)\right]_0^1 + \int_0^1 u'(x)v'(x) = 0$$

$$\int_0^1 u'(x)\, v'(x)\, dx = u'(1)\underbrace{v(1)}_{=0} - \underbrace{u'(0)}_{=5} v(0)$$

$\forall v \in V$

FEM Find $U \in V_h$, $U = \sum_{i=0}^{N} \xi_i \varphi_i(x)$, s.t:

$$\int_0^1 U'(x)\varphi_j'(x)\, dx = -5\varphi_j(0) \qquad j = 0, \ldots, N$$

$V_h = \{ v \in V,\ \text{p.w. linear} \}$

$$\sum_0^N \xi_i \underbrace{\int_0^1 \varphi_i'(x)\,\varphi_j'(x)}_{a_{ji}} = \underbrace{-5\,\varphi_j(0)}_{:=b_j} \qquad j=0,\dots,N$$

$\underline{\varphi_0,\ \varphi_1,\ \varphi_2}$

$$b = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$$

$$a_{00} = \int_0^h \left(-\frac{1}{h}\right)^2 = \dots = \frac{1}{h}$$

$$a_{ii} = \dots = \frac{2}{h}$$

$$a_{i,i+1} = a_{i+1,i} = \dots = -\frac{1}{h}$$

$$\Rightarrow \frac{1}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$$

b) Solving this yields $\begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{bmatrix} = -h\begin{bmatrix} 15 \\ 10 \\ 5 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 15 \\ 10 \\ 5 \end{bmatrix}$

$$u(x) = -15h\,\varphi_0(x) - 10h\,\varphi_1(x) - 5h\,\varphi_2(x)$$

on $0 \le x \le h$: $\quad -15h\varphi_0 - 10h\varphi_1 = \dots = 5x-5$

on $h \le x \le 2h$: $\quad -10h\varphi_1 - 5h\varphi_2 = \dots = 5x-5$

on $2h \le x \le 1$: $\quad -5h\varphi_2 = \dots = 5x-5$

$\underline{U(x) = 5x-5}$ Same as exact solution

5.17) Prove an a priori error estimate for the FEM for the problem

$$\begin{cases} -u''(x) + u'(x) = f(x) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

Solution

VF: Multiply by $v \in H_0^1$ and integrate $\Rightarrow$

Find $u \in H_0^1(0,1)$:

$$\underbrace{\int_0^1 u'v' + \int_0^1 u'v}_{:= a(u,v)} = \underbrace{\int_0^1 fv}_{=: F(v)}$$

Note that $a(u,v)$ is not symmetric. $\int_0^1 u'v = -\int_0^1 uv'$

if $u=v$, then $\int_0^1 u'u = 0$.

Let $u_h \in V_0^h$ be the solution to $a(u_h, v) = F(v) \ \forall v \in V_0^h$

$$a(u-u_h, u-u_h) = \int_0^1 (u-u_h)'(u-u_h)' + \underbrace{\int_0^1 (u-u_h)'(u-u_h)}_{=0} =$$

$$= \|(u-u_h)'\|^2_{L^2(0,1)}$$

$$\|(u-u_h)'\|^2_{L^2(0,1)} = a(u-u_h, u-u_h) = \left\{ \begin{matrix} v \in V_0^h \\ \text{arbitrary} \end{matrix} \right\} =$$

$$= a(u-u_h, u-v+\underbrace{v-u_h}) = \left\{ \begin{matrix} \text{Galerkin Orthogonality:} \\ a(u-u_h, v) = 0 \quad \forall v \in V_0^h \end{matrix} \right\} =$$

$$= a(u-u_h, u-v) = \int_0^1 (u-u_h)'\left((u-v)' + (u-v)\right) \leq$$

$$\leq \{C-S\} \leq \|(u-u_h)'\|_{L^2(0,1)} \|(u-v)' + (u-v)\|_{L^2(0,1)} \leq$$

$$\leq \{\Delta\text{-ineq.}\} \leq \|(u-u_h)'\|_{L^2(0,1)} \|(u-v)'\|_{L^2(0,1)} + \|(u-u_h)'\|_{L^2(0,1)} \|u-v\|_{L^2(0,1)}$$

$$\leq \left\{ \begin{matrix} u-v \in H_0^1 \\ \text{Poincaré:} \\ \|u-v\|_{L^2} \leq \|(u-v)'\|_{L^2} \end{matrix} \right\} \leq 2\|(u-u_h)'\|_{L^2(0,1)} \|(u-v)'\|_{L^2(0,1)}$$

$$\therefore \quad \|(u-u_h)'\|_{L^2(0,1)} \leq 2\|(u-v)'\|_{L^2(0,1)}$$

Take $v = \pi_1 u$

$$\|(u-u_h)'\|_{L^2(0,1)} \leq 2\|u' - (\pi_1 u)'\|_{L^2(0,1)} \leq \{\text{thm } 3.2\} \leq$$

$$\leq C\|u''\|_{L^2(0,1)}.$$

$\%$

8) Prove that specifying the information $p(a), p'(a)$, $p(b)$ and $p'(b)$ suffices to determine polynomials in $\mathcal{P}^{(3)}(a,b)$ uniquely.

Solution A polyn. in $\mathcal{P}^{(3)}$ can be written

$$p = c_1 x^3 + c_2 x^2 + c_3 x + c_4$$

$$p' = 3c_1 x^2 + 2c_2 x + c_3$$

Unknowns: c_1, c_2, c_3, c_4

$$\Rightarrow \begin{bmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ 3a^2 & 2a & 1 & 0 \\ 3b^2 & 2b & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} p(a) \\ p(b) \\ p'(a) \\ p'(b) \end{bmatrix}$$

This lin syst. has a unique sol if $\det A \neq 0$

$\det A = \dots = -(b-a)^4 \neq 0$, $b \neq a$. ⁄.

12) An elastic string on $(0,1)$ with spring coefficient $c(x)>0$ can be modelled as

$$\begin{cases} -u''(x) + c(x)u(x) = f(x), & 0<x<1 \\ u(0) = u(1) = 0 \end{cases}$$

Formulate a cG(1) method for this problem. Compute stiffness matrix when c is constant. Is the stiffness matrix still symmetric, positive definite and tridiagonal?

Solution

VF: Multiply by $v \in H_0^1$ and integrate:

$$(*) \quad \int_0^1 u'(x)v'(x) + \int_0^1 c(x)u(x)v(x) = \int_0^1 f(x)v(x)$$

Exchanging roles of u and v doesn't change anything
$\Rightarrow$ Stiffness matrix will be symmetric.

Let $\{x_i\}_{i=0}^{m+1}$ be a partition of $[0,1]$ with equidistant step size h. Let $\{\varphi_i\}_{i=1}^{m}$ be hat functions, and let $u_h = \sum_{i=1}^{m} \xi_i \varphi_i(x)$, $v(x) \in \text{span}\{\varphi_i\}_{i=1}^{m}$. Then $(*)$ becomes

$$\sum_{i=1}^{m} \xi_i \underbrace{\int_0^1 \left(\varphi_i'(x)\varphi_j'(x) + c(x)\varphi_i(x)\varphi_j(x)\right)}_{a_{ji}} = \int_0^1 f(x)\varphi_j(x)$$

$j=1,\dots,m$

φ_i and φ_j have common support when $|i-j|\le 1$, so A will be tridiagonal.

A when c is constant?

$$a_{ii} = \int_0^1 (\varphi_i'(x))^2 dx + c\int_0^1 (\varphi_i(x))^2 dx = \dots = \frac{2}{h} + \frac{2ch}{3}$$

$$a_{i,i+1} = a_{i+1,i} = \int_0^1 \varphi_i'(x)\varphi_{i+1}'(x)\,dx + c\int_0^1 \varphi_i(x)\varphi_{i+1}(x)\,dx =$$

$$= \dots = -\frac{1}{h} + \frac{ch}{6}$$

Is A positive definite?

Pos. def. if $v^T A v > 0 \quad \forall v \in \mathbb{R}^m \setminus \{0\}$

$$A = S + cM = \left((\varphi_i', \varphi_j') + c(\varphi_i, \varphi_j) \right)_{i,j=1}^{m}$$

$$v^T A v = \sum_{i=1}^{m} \sum_{j=1}^{m} v_i \left((\varphi_i', \varphi_j') + c(\varphi_i, \varphi_j) \right) v_j =$$

$$= \sum \sum \left(v_i (\varphi_i', \varphi_j') v_j + c\, v_i (\varphi_i, \varphi_j) v_j \right) =$$

$$= \left(\sum_i v_i \varphi_i', \sum_j v_j \varphi_j' \right) + c \left(\sum_i v_i \varphi_i, \sum_j v_j \varphi_j \right) =$$

$$= \| w' \|^2_{L^2(0,1)} + c \| w \|^2_{L^2(0,1)} > 0 \quad \text{if } c > 0$$

with $w = \sum_i v_i \varphi_i \in V_n \setminus \{0\}$

Thus positive definite.

$\checkmark$

11) Compute the stiffness matrix and load vector for the $cG(1)$ method on a uniform triangulation for

$$\begin{cases} -(a(x)u'(x))' = f(x) & 0<x<1 \\ u(0) = u(1) = 0 \end{cases}$$

with $a(x) = 1+x$ and $f(x) = \sin(x)$

Solution. Variational formulation:

multiply by a test function $v(x) \in H_0^1(0,1)$ and integrate:

$$-\int_0^1 ((1+x)u'(x))' v(x)\,dx = \int_0^1 \sin(x)\, v(x)\,dx$$

$$\text{P.I} \Rightarrow \int_0^1 (1+x)u'(x)v'(x)\,dx = \int_0^1 \sin(x)\,v(x)\,dx \quad (*)$$

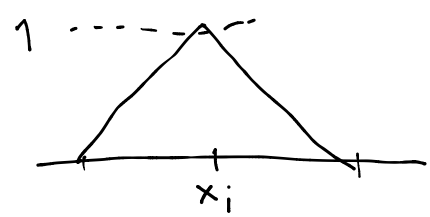
$v(0)=v(1)=0$

Let $\{x_i\}_{i=0}^{m+1}$ be a partition of $[0,1]$ with equidistant step size h.

CG(1) method: $\quad u_h(x) = \sum_{i=1}^{m} \xi_i \varphi_i(x)$

$\qquad v(x) \in \operatorname{span}\left(\{\varphi_j\}_{j=1}^{m}\right)$

$$\varphi_i(x) = \begin{cases} \dfrac{x-x_{i-1}}{h} & x\in[x_{i-1},x_i] \\ \dfrac{x_{i+1}-x}{h} & x\in[x_i,x_{i+1}] \\ 0 & \text{else} \end{cases}$$



$$\varphi_i'(x) = \begin{cases} 1/h & x\in(x_{i-1},x_i) \\ -1/h & x\in(x_i,x_{i+1}) \\ 0 & \text{else} \end{cases}$$

$$(*) \Rightarrow \int_0^1 (1+x) \sum_{i=1}^{m} \xi_i \varphi_i'(x)\varphi_j'(x)\,dx = \int_0^1 \sin(x)\varphi_j(x)\,dx$$

$$\Rightarrow \sum_{i=1}^{m} \xi_i \underbrace{\int_0^1 (1+x)\varphi_i'(x)\varphi_j'(x)\,dx}_{a_{ij}} = \underbrace{\int_0^1 \sin(x)\varphi_j(x)}_{b_j}, \quad j=1,\dots,m$$

<u>stiffness matrix:</u>

<u>$i=j$</u> $\quad a_{ii} = \int_0^1 (1+x)\left(\varphi_i'(x)\right)^2 =$

$$= \int_{x_{i-1}}^{x_i} (1+x)\frac{1}{h^2} + \int_{x_i}^{x_{i+1}} (1+x)\frac{1}{h^2} = \dots =$$

$$= \frac{2}{h} + \frac{2x_i}{h} = \{x_i = ih\} = \underline{\frac{2}{h} + 2i}$$

$\underline{j=i+1}$ $$a_{i,i+1} = a_{i+1,i} = \int_0^1 (1+x)\,\varphi_i'(x)\,\varphi_{i+1}'(x)\,dx$$

$$= -\int_{x_i}^{x_{i+1}} (1+x)\frac{1}{h^2}\,dx = \dots = \underline{-\frac{1}{h} - \frac{2i+1}{2}}$$

$$\Rightarrow A = (a_{ij})_{i,j=1,\dots,m} = \begin{bmatrix} \frac{2}{h}+2 & -\frac{1}{h}-\frac{2+1}{2} & 0 & 0 & \cdots \\ -\frac{1}{h}-\frac{2+1}{2} & \frac{2}{h}+4 & -\frac{1}{h}-\frac{4+1}{2} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \end{bmatrix}$$

Load vector

$$b_j = \int_0^1 \sin(x)\,\varphi_j(x)\,dx =$$

$$= \int_{x_{j-1}}^{x_j} \frac{\sin(x)(x-x_{j-1})}{h} + \int_{x_j}^{x_{j+1}} \frac{\sin(x)(x_{j+1}-x)}{h} = \underset{PI}{\dots}$$

$$= \frac{1}{h}\left(2\sin(x_i) - \sin(x_{i-1}) - \sin(x_{i+1})\right)$$

$$x_i = ih$$

$$b = \frac{1}{h}\begin{bmatrix} 2\sin(h) - \sin(0\cdot h) - \sin(2h) \\ 2\sin(2h) - \sin(h) - \sin(3h) \\ \vdots \end{bmatrix}$$

//.