**Problems in PDE1 (mostly from CDE, by Estep el al)**

**Galerkin's Method**

1. Let $V^{(q)} = \mathcal{P}^q(0,1)$ be the set of polynomials of degree $\leq q$. Define
$$V_0^{(q)} := \{v : \ v \in V^{(q)}, \ v(0) = 0\}.$$
Prove that $V_0^{(q)}$ is a subspace of $V^{(q)}$.

2. Consider the initial value problem
$$\begin{cases} \dot{u}(t) = \lambda u(t) & 0 < t \leq 1, \\ u(0) = u_0. \end{cases} \tag{1}$$
Compute the Galerkin approximation for $q = 1, 2, 3$ and 4 assuming that $u_0 = \lambda = 1$.

3. Compute the $L_2(0,1)$ projection into $\mathcal{P}^3(0,1)$ of the exact solution $u$ for (1) and compare it with the Galerkin's solution $U$.

4. Consider the boundary value problem
$$\begin{cases} -(a(x)u'(x))' = f(x) & 0 < x \leq 1, \\ u(0) = u(1) = 0. \end{cases} \tag{2}$$
Let $a \equiv 1$ and $f = x$. Use the uniform mesh with $h = 1/4$, and compute the Galerkin approximation for $u$.

5. Prove that $V^{(q)} = \text{span}\{\sin(i\pi x), 1 \leq i \leq q\}$ is a subspace of the continuous functions on $[0,1]$ that satisfy the vanishing boundary conditions and show that $\{\sin(i\pi x)\}_{i=1}^q$ is an orthogonal basis for $V^{(q)}$ with respect to the $L_2$ inner product.

**Polynomial interpolation**

6. Prove the following $L_p(a,b)$ error estimates for the interpolation, with $p = 1$ and 2:
$$\|f - \pi_1 f\|_{L_p(a,b)} \leq (b-a)^2 \|f''\|_{L_p(a,b)}. \tag{3}$$

7. Write down a basis for the set of piecewise quadratic polynomials $W_h^{(2)}$ on $(a,b)$ and plot a sample of the functions.

8. Prove that specifying the information $p(a)$, $p'(a)$, $p(b)$, and $p'(a)$ suffices to determine polynomials $\mathcal{P}^3(a,b)$ uniquely.

9. Let $I = (a,b)$ and consider a partition of $I$ as $\{x_i\}_{i=0}^{m+1}$ of $(a,b)$ with mesh function $h(x) = h_i = (x_i - x_{i-1})$. Prove that any value

of $f$ on the subinterval can be used to define $\pi_h f$ satisfying the error bound:

$$\|f - \pi_h f\|_{L_\infty(a,b)} \le \max_{1\le i\le m+1} h_i \|f'\|_{L_\infty(I_i)} = \|hf'\|_{L_\infty(a,b)}. \tag{4}$$

Prove that choosing the midpoint improves the bound by an extra factor 1/2.

10. Construct a piecewise cubic polynomial function with a continuous first derivative that interpolates a function and its first derivative at the nodes of a partition.

**Two-point boundary value problems**

11. Compute the stiffness matrix and load vector for the cG(1) method on a uniform triangulation for the problem (2) with $a(x) = 1 + x$ and $f(x) = \sin(x)$.

12. An elastic string on $(0,1)$ with spring coefficient $c(x) > 0$ can be modeled as

$$\begin{cases} -u''(x) + c(x)u(x) = f(x), & 0 < x \le 1, \\ u(0) = u(1) = 0. \end{cases} \tag{5}$$

Formulate a cG(1) method for (5). Compute the stiffness matrix when $c$ is a constant. Is the stiffness matrix still symmetric, positive-definite, and tridiagonal?

13. Show that the minimization problem for (2): Find $U \in V_h$ such that if

$$F(U) \le F(v), \quad \forall v \in V_h,$$

where

$$F(w) = \frac{1}{2}\int_0^1 a(w')^2\,dx - \int_0^1 fw\,dx,$$

takes the matrix form: Find $\chi = (\chi_j) \in \mathbb{R}^M$ that minimizes the quadratic function $\frac{1}{2}\eta^T A\eta - b^T\eta$ for $\eta \in \mathbb{R}^M$.

14. Consider the Neumann problem

$$\begin{cases} -(a(x)u'(x))' = f(x) & 0 < x \le 1, \\ u(0) = 0, \quad a(1)u'(1) = g_1, \end{cases} \tag{6}$$

The discrete variational formulation is formulated as: Find $U \in V_h$, (where $V_h$ is the space of continuous functions $v$ that are piecewise linear with respect to a partition $\mathcal{T}_h$ on $(0,1)$: $(\{x_i\}_{i=0}^{M+1})$ such that $v(0) = 0$), such that

$$-g_1 v(1) + \int_0^1 aU'v'\,dx = \int_0^1 fv\,dx, \qquad \forall v \in V_h. \tag{7}$$

Compute the coefficients of the stiffness matrix for the finite element method (7) using a uniform partition and assuming $a = f = g_1 = 1$. Check if the discrete equation corresponding to the test function $\varphi_{M+1}$ at $x = 1$ looks like a discrete analogue of the Neumann condition.

15. Consider the Robin problem

$$\begin{cases} -(a(x)u'(x))' = f(x) & 0 < x \leq 1, \\ u(0) = 0, \quad a(1)u'(1) + \gamma(u(1) - u_1) = g_1, \end{cases} \tag{8}$$

where $\gamma > 0$ is a given boundary heat conductivity, $u_1$ is a given outside temperature and $g_1$ a given heat flux. Show that the variational formulation to this problem is given as

$$\int_0^1 au'v'\,dx + \gamma u(1)v(1) = \int_0^1 fv\,dx + g_1 v(1) + \gamma u_1 v(1), \qquad \forall v \in V. \tag{9}$$

$$V = \{v : \int_0^1 (v(x)^2 + v'(x))^2\,dx < \infty, \quad v(0) = 0\}.$$

16. Use the trapezoidal rule to evaluate integrals involving $a$ and $f$ and recompute the coefficients of $A$ and $b$ in problem 11.

17. Prove the following inequality for functions $v(x)$ on $I = (0,1)$ with $v(0) = 0$,

$$v(y) = \int_0^y v'(x)\,dx \leq \Big(\int_0^y a^{-1}\,dx\Big)^{1/2}\Big(\int_0^y a(v')^2\,dx\Big)^{1/2}, \quad y \in I, \tag{10}$$

where $a(x) > 0$ and $a^{-1}$ denotes the multiplicative inverse of $a$, i.e. $a^{-1} = 1/a$. Use this inequality to show that if $a^{-1}$ is integrable on $I = (0,1)$, that is, $\int_I a^{-1}\,dx < \infty$, then a function $v$ is small in the supremum norm on $[0,1]$ if $\|v\|_E$ is small and $v(0) = 0$.

18. Prove an a priori and an a posteriori error estimate for the cG(1) method applied to the boundary value problem

$$\begin{aligned} -u'' + bu' + u &= f \quad \text{in } (0,1), \\ u(0) = u(1) &= 0. \end{aligned} \tag{11}$$

19. Consider the problem (11)

a) with non-homogeneous Dirichlet boundary conditions: $u(0) = u_0$ and $u(1) = u_1$.

b) with the mixed boundary conditions: $u(0) = u_0$ and $u'(1) = g_1$.

20. Consider the problem

$$-\epsilon u'' + xu' + u = f \quad \text{in } I = (0,1), \quad u(0) = u'(1) = 0,$$

where $\epsilon$ is a positive constant, and $f \in L_2(I)$. Prove that

$$\|\epsilon u''\| \leq \|f\|,$$

where $\|\cdot\|$ is the $L_2(I)$-norm.

**Scalar initial value problem**

21. Consider the initial value problem

$$\begin{cases} \dot{u}(t) + a(t)u(t) = f(t) & 0 < t \le T, \\ u(0) = u_0. \end{cases} \tag{12}$$

Assume that

$$\int_{I_j} f(s)\,ds = 0, \quad \text{for } j = 0, 1, \ldots, \tag{13}$$

where $I_j = (t_{j-1}, t_j)$, $t_j = jk$ with $k$ a positive constant. Prove that if $a(t) \ge 0$, then the solution to (13) satisfies

$$|u(t)| \le e^{-A(t)}|u_0| + \max_{0 \le s \le t} |kf(s)|. \tag{14}$$

22. Compute the cG(1) approximation for the initial boundary value problem (12) for

a) $a(t) = 4$, with $f(t) = t^2$ and $u_0 = 1$.

b) $a(t) = -t$, with $f(t) = t^2$ and $u_0 = 1$.

In each case determine the condition on the step size that guarantees that $U$ exists.

23. Consider the discontinuous Galerkin method dG(0) for the equation (12) in the case $f = 0$ , $a \ge 0$. Prove the following stability estimate

$$|U_N|^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \le |u_0|^2.$$

24. Recall the stability factors

$$S(t_N) = \Big(\int_0^{t_N} |\dot{\varphi}| dt\Big)/|e_N|, \quad \text{and} \quad \tilde{S}(t_N) = \Big(\int_0^{t_N} |\varphi| dt\Big)/|e_N|.$$

for the dual problem

$$-\dot{\varphi} + a\varphi = 0, \quad \text{for} \quad t_N > t \ge 0, \quad \varphi(t_N) = e_N. \tag{15}$$

Prove that

$$\tilde{S}(t_N) \le t_N(1 + S(t_N)).$$

25. Assume that $a > 0$ is constant. Prove that, if $a$ is small, then

$$\tilde{S}(t_N) >> S(t_N).$$

**Calculus and piecewise polynomials in several dimensions**

26. If $u(x_1, x_2) = (u_1, u_2)^T$ is a vector function, then $rot\, u$ is the scalar function
$$rot\, u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},$$
while if $u$ is scalar, then $rot\ u$ is the vector function
$$rot\, u = \Big(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}\Big),$$
Following identities follow directly from these definitions.
$$\text{div rot} u = 0, \qquad \text{rot grad } u = 0. \tag{16}$$
Prove that in two dimensions and with $u$ scalar
$$rot\, rot\, u = -\Delta u.$$

27. Prove that $-\Delta u(x) = 0$, for $x \neq 0$ if $u : \mathbb{R}^2 \to \mathbb{R}$ is given by $u(x) = \log(|x|^{-1})$.

28. Prove (16) for the functions $u$ defined in $\mathbb{R}^3$, i.e. for $u : \mathbb{R}^3 \to \mathbb{R}^3$.

29. Let $u : \mathbb{R}^3 \to \mathbb{R}$ have a local minimum at $y \in \mathbb{R}^3$, that is $u(y) \leq u(x)$ for all $x$ satisfying $|x - y| \leq \delta$ for some $\delta > 0$. Prove that $\nabla u(y) = 0$.

30. Consider the Laplacian in the polar coordinates: set
$$x_1 = r\cos(\theta), \qquad x_1 = r\sin(\theta),$$
then
$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{17}$$
Show using (17) that the function $u = c_1 \log(r) + c_2$ where $c_i$ are arbitrary constants, is a solution of the Laplace equation $\Delta u(x) = 0$ in $\mathbb{R}^2$ for $x \neq 0$. Are there other solutions of the Laplace equation in $\mathbb{R}^2$ which are invariant (i.e. it depends only on $r = |x|$)?

31. Prove that the function $u = \frac{c_1}{r} + c_2$, where $c_i$ are arbitrary constants, is a solution of Laplace's equation $\Delta u(x) = 0$ in $\mathbb{R}^3$ for $x \neq 0$.

32. To construct a set of basis functions (in 2D) for $V_h$, we begin by describing a set of *element basis functions* for triangles. Assuming that a triangle has nodes at $\{a^1, a^2, a^3\}$, the element nodal basis is the set of functions $\lambda_i \in \mathcal{P}^1(K),\ i = 1, 2, 3$, such that
$$\lambda_i(a^j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$
compute explicit formulas for the $\lambda_i$.

33. Prove that if $K_1$ and $K_2$ are neighboring triangles and $w_1 \in \mathcal{P}^2(K_1)$ and $w_2 \in \mathcal{P}^2(K_2)$ agree at the three nodes on the common boundary, then $w_1 \equiv w_2$ on the common boundary.

**The Poisson Equation**

34. If

$$-\Delta E = \delta_0, \quad \text{in} \quad \mathbb{R}^d, \tag{18}$$

where $\delta_0$ is the Dirac $\delta$ distribution, then $E$ is called the *fundamental solution* of $-\Delta$ in $\mathbb{R}^d$. Equivalently, for any smooth test function $v$ vanishing outside a bounded set, $E$ satisfies

$$-\int_{\mathbb{R}^d} \Delta E(x) v(x) = v(0). \tag{19}$$

Prove that

$$E(x) = \frac{1}{2\pi} \log(\frac{1}{|x|}) \tag{20}$$

is a fundamental solution of $-\Delta$ in $\mathbb{R}^2$.

35. Consider the Poisson equation with homogeneous Dirichlet boundary condition

$$\begin{cases} -\Delta u(x) = f(x), & \text{for } x \in \Omega \\ u(x) = 0, & \text{for } x \in \Gamma, \end{cases} \tag{21}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with polygonal boundary $\Gamma$. The variational formulation for (21) is given by

$$(\nabla u, \nabla v) = (f, v), \qquad \forall v \in V, \tag{22}$$

where

$$V = \{v : \int_\Omega (|\nabla v|^2 + v^2)\, dx < \infty, \quad v = 0, \quad \text{on } \Gamma\}. \tag{23}$$

Prove that the set of functions that are continuous and piecewise differentiable on $\Omega$ and vanish on $\Gamma$, is a subspace of $V$

36. Assume that the solution of the problem (22) is continuous on $\Omega \cup \Gamma$, show that it is unique.

37. Describe the discrete system of equations for a a piecewise polynomial approximation for (21) with general continuous polynomial basis functions $\{\varphi_j\}_{j=1}^M$. Show that the stiffness matrix for this approximation is symmetric and positive definite.

38. We define the $L_2$-projection $P_h v$ of a function $v \in L_2(\Omega)$ into the finite element space $V_h$ by

$$(P_h v, w) = (v, w), \quad \forall w \in V_h. \tag{24}$$

We also define the discrete Laplacian $\Delta_h$ by

$$-(\Delta_h w, v) = (\nabla w, \nabla v), \quad \forall v \in V_h. \tag{25}$$

Verify that we may express the finite problem

$$(\nabla U, \nabla v) = (f, v), \qquad \forall v \in V_h, \tag{26}$$

as finding $U \in V_h$ such that

$$-\Delta_h U = P_h f. \tag{27}$$

39. Substituting $P_h v = \sum_j \eta_j \varphi_j$ and choosing $w = \varphi_i$, $i = 1, 2, \ldots, M$, we obtain the linear system

$$M\eta = b, \tag{28}$$

where $M$ is the mass matrix with the coefficients $(\varphi_j, \varphi_i)$ and $b$ is the load vector with components $(v, \varphi_i)$. Prove that the mass matrix $M$ is symmetric and positive definite.

40. Consider the Poisson equation with non-homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u(x) = f(x), & \text{for } x \in \Omega \\ u(x) = g(x), & \text{for } x \in \Gamma, \end{cases} \tag{29}$$

where $g$ is a given boundary data. The variational formulation takes the following form: find $u \in V_g$, where

$$V_g = \{v : v = g \text{ on } \Gamma \text{ and } \int_\Omega (|\nabla v|^2 + v^2)\, dx < \infty\},$$

such that

$$(\nabla u, \nabla v) = (f, v), \qquad \forall v \in V_0, \tag{30}$$

with

$$V_0 = \{v : v = 0 \text{ on } \Gamma \text{ and } \int_\Omega (|\nabla v|^2 + v^2)\, dx < \infty\}.$$

Show that $v_g$ is not a vector space. Prove that the solution of the weak problem (30) is unique.

41. Compute the discrete system of equations for the finite element approximation of $-\Delta u = 1$ in $\Omega = (0,1) \times (0,1)$ with $u = 0$ on the side $x_1 = 0$, $u = x_1$ for $x_2 = 0$, $u = 1$ for $x_1 = 1$ and $u = x_1$ for $x_2 = 1$, using the standard triangulation.

42. Compute the discrete system of equations for the problem $-\Delta u = 1$ in $\Omega = (0,1)\times(0,1)$ with $u = 0$ on the side $x_2 = 0$ and $\partial_n u + u = 1$ on the other three sides of $\Omega$ using the standard triangulation. Note the contribution to the stiffness matrix from the nodes on the nodes on the boundary.

**Heat, and wave equations**

43. Consider the homogeneous heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(\cdot,0) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases} \tag{31}$$

Show that, for $t > 0$, the fundamental solution (solution to the problem (31) with $u_0 = \delta_0$) is given by

$$F(x,t) = \frac{1}{4\pi t} \exp\Big(-\frac{|x|^2}{4t}\Big). \tag{32}$$

44. Verify that the solution of (31) is given by

$$u(x,t) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} u_0(y) \exp\Big(-\frac{|x-y|^2}{4t}\Big)\, dy. \tag{33}$$

45. Consider the non-homogeneous heat equation

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t) & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{for } x \in \Gamma, \quad 0 < t \le T, \\ u(\cdot,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{34}$$

where $\Omega \subset \mathbb{R}^2$. Formulate the cG(1)dG(0) finite element method for the heat equation (34), using the lumped mass quadrature rule in space and the two point Gauss quadrature rule for the time integral over $I_n$.

46. Consider the non-homogeneous heat equation in 1D with variable coefficient:

$$\begin{cases} u_t(x,t) - (a(x,t)u'(x,t))' = f(x,t) & (x,t) \in (0,1) \times \times (0,\infty), \\ u(0,t) = u(1,t) = 0, & t \in (0,\infty), \\ u(x,0) = u_0(x) & x \in (0,1). \end{cases} \tag{35}$$

Formulate a cG(1)dG(0) finite element method for this problem.

47. Consider the homogeneous heat equation (31). Show that under the assumption $k_n \le C k_{n-1}$ on time step. the cG(1)dG(1) solution $U_n$ satisfies

$$\|U_N^-\| \le \|U_0^-\|, \qquad 1 \le n \le N.$$

48. Consider the non-homogeneous wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t) & (x,t) \in \mathbb{R} \times \times (0,\infty), \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = \dot{u}_0(x) & x \in \mathbb{R}. \end{cases} \tag{36}$$

Prove the extension of the d'Alembert formula for (36):

$$u(x,t) = \frac{1}{2}[u_0(x+t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \dot{u}_0(y)\, dy + \frac{1}{2}\int_{\Delta} f(y,s)\, dy\, ds \tag{37}$$

where

$$\Delta = \{(y, s) : |x - y| \leq t - s,\ s \geq 0\},$$

denotes the *triangle of dependence*.

49. Verify (using d'Alemberts formula) the formula for solution for $\ddot{u} = c^2 u''$ for $x > 0$, $t > 0$, $u(x, 0) = u_0(x)$ and $\dot{u}(x, 0) = v_0(x)$ for $x > 0$, and a) $u(0, t) = 0$, b) $u(0, t) = g(t)$ for $t > 0$.

50. Prove the conservation of energy for the homogeneous wave equation in a two dimensional domain $\Omega$:

$$\|\dot{u}(\cdot, t)\|^2 + \|\nabla u(\cdot, t)\|^2 = \|\dot{u}_0\|^2 + \|\nabla u_0\|^2, \qquad t > 0.$$

51. Consider the wave equation

$$\begin{cases} \ddot{u} - \Delta u = f & (x, t) \in \Omega \times (0, \infty), \\ u = 0, & (x, t) \in \Gamma \times (0, \infty), \\ u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = \dot{u}_0(x) & x \in \Omega. \end{cases} \tag{38}$$

52. Show that for any proper subdomain $\omega$ of $\Omega$ and $t > 0$ such that $\omega(t) \subset \Omega$, the solution $u$ of the homogeneous wave equation satisfies

$$\|\dot{u}(\cdot, t)\|^2_{L_2(\omega)} + \|\nabla u(\cdot, t)\|^2_{L_2(\omega)} \leq \|\dot{u}_0\|^2_{L_2(\omega(t))} + \|\nabla u_0\|^2_{L_2(\omega(t))}.$$

**Riesz and Lax-Milgram Theorems**

53. Let $B = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disc in $\mathbb{R}^2$. Find condition on $r$ for which $|x|^r \in H^1(B)$ but $|x|^s \notin H^1(B)$ for any $s < r$.

54. Define $H^2(B)$ and find a function that is in $H^1(B)$ but not in $H^2(B)$ where $B$ is the unit disc.

55. Prove the *Poincare-Friedrichs* inequality:

$$\|v\|^2_{L_2(\Omega)} \le C_\Omega\Big(\|v\|^2_{L_2(\Gamma)} + \|\nabla v\|^2_{L_2(\Omega)}\Big), \qquad \forall v \in H^1(\Omega). \tag{39}$$

56. Verify the *trace theorem:* If $\Omega$ is a bounded domain with boundary $\Gamma$, then there is constant $C$ such that

$$\|v\|_{L_2(\Gamma)} \le C\|v\|_{H^1(\Omega)}, \qquad \forall v \in H^1(\Omega). \tag{40}$$

57. Show that there is no constant $C$ such that $\|v\|_{L_2(\Gamma)} \le C\|v\|_{L_2(\Omega)}$, $\forall v \in L_2(\Omega)$.

58. Consider the convection-diffusion problem

$$-div(\varepsilon\nabla u + \beta u) = f, \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0, \text{ on } \quad \partial\Omega, \qquad u \in H^1_0(\Omega),$$

where $\Omega$ is a bounded convex polygonal domain, $\varepsilon > 0$ is constant, $\beta = (\beta_1(x), \beta_2(x))$ and $f = f(x)$. Determine the conditions in the Lax-Milgram theorem that would guarantee existence of a unique solution for this problem. Prove a stability estimate for $u$ i terms of $||f||_{L_2(\Omega)}$, $\varepsilon$ and $diam(\Omega)$, and under the conditions that you derived.

59. Consider the following boundary value problem (Robin boundary condition and a convex bounded domain) in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$,

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + u = g, & \text{on } \Gamma = \partial\Omega. \end{cases}$$

a) Prove the $L_2$ stability estimate

$$||\nabla u||^2_{L_2(\Omega)} + \frac{1}{2}||u||^2_{L_2(\Gamma)} \le \frac{1}{2}||g||_{L_2(\Gamma)}.$$

b) Verify the conditions on Riesz/Lax-Milgram theorem for this problem, i.e., prove V-ellipticity and continuity of the *corresponding* bilinear form as well as the continuity of the *corresponding* linear form.

60. Verify the conditions on Riesz/Lax-Milgram theorem for the problem

$$d^4u/dx^4 = f, \quad x \in (0, 1),$$

with $u(0) = u'(0) = u(1) = u'(1) = 0$.

## Answers

### Galerkin's Methods

2.

$q = 1 \Longrightarrow u(t) = 1 + 3t,$

$q = 2 \Longrightarrow u(t) = 1 + \frac{8}{11}t + \frac{10}{11}t^2,$

$q = 3 \Longrightarrow u(t) = 1 + \frac{30}{29}t + \frac{45}{116}t^2 + \frac{35}{116}t^3,$

$q = 4 \Longrightarrow u(t) \approx 1{+}0.9971t{+}0.5161t^2{+}0.1311t^3{+}0.0737t^4.$

3. $Pu(t) \approx 0.9991 + 1.0183t + 0.4212t^2 + 0.2786t^3.$

4. Exact solution $u(x) = \frac{1}{6}(-x^3+x)$, FEM- solution at nod points: $U(x_1) = 0.0391,\ U(x_2) = 0.0625,\ U(x_3) = 0.0547.$

### Polynomial interpolation

7. For example we may choose the following basis:

$$\varphi_{i,j}(x) = \begin{cases} 0, & x \notin [x_{i-1}, x_i], \\ \lambda_{i,j}(x), & x \in [x_{i-1}, x_i], \end{cases} \quad i = 1, \dots m{+}1,\ \ j = 0, 1, 2.$$

where for $\chi_i \in (x_{i-1}, x_i)$

$$\begin{cases} \lambda_{i,0}(x) = \frac{(x-\chi_i)(x-x_i)}{(x_{i-1}-\chi_i)(x_{i-1}-x_i)}, \\ \lambda_{i,1}(x) = \frac{(x-x_{i-1})(x-x_i)}{(\chi_i-x_{i-1})(\chi_i-x_i)}, \\ \lambda_{i,2}(x) = \frac{(x-x_{i-1})(x-\chi_i)}{(x_i-x_{i-1})(x_i-\chi_i)}. \end{cases}$$

### Two-points boundary value problem

11. $a_{ii} = \frac{2}{h} + 2i, \quad a_{i-1,ii} = -\frac{1}{h} - i - \frac{1}{2},$

$b_i = \frac{1}{h}\Big(2\sin(hi) - \sin(h(i-1)) - sin(h(i+1))\Big).$

12. $a_{ij} = \tilde{a}_{ij} + c\int_0^1 \varphi_j(x)\varphi_i(x)\,dx, \quad \tilde{a}_{ij} = \int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx,$

$b_i = \int_0^1 f(x)\varphi_i(x)\,dx.$

The coefficient matrix is symmetric, positive definite and tridiagonal. It can even be diagonal, depending on the choice of $c$ (if we choose $c = \frac{6}{h_{i+1}}$).

14. For $i, j = 1, \ldots, M$ we have as usual $a_{ii} = \frac{2}{h}, \quad a_{i,i+1} = -\frac{1}{h}$. This also holds for $A_{M,M+1} = A_{M+1,M}$, but since $\varphi_{M+1}$ has support only to the left of $x_{M+1}, a_{M+1,M+1} = \frac{1}{h}$.

18. A priori: $\|e\|_E \leq \|u - v\|_E(1 + b)$.

A posteriori: $\|e\|_E \leq C_i \|hR(U)\|$,

$R(U) = f + U'' - bU' - U$.

## Scalar initial value problem

22. Hint, Use the algorithm on page 145 of Lecture Notes.

## Calculus and piecewise ploynomials in several dimensions

32. $\lambda_1(x) = 1 - D^{-1}(a^3 - a^2)^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (x - a^1)$, with

$D = (a_1^2 - a_1^1)(a_2^3 - a_2^1) - (a_2^3 - a_2^1)(a_1^3 - a_1^1)$.

## The Poisson equation

42. You get block-diagonal coefficient matrix $A$.

$b = (6 + 3h, 6h, \ldots, 6h, 6 + 3h|, ..., |6 + 3h, 6h, \ldots, 6h, 6 + 3h, |6 + 2h, 6 + 3h, \ldots, 6 + 3h, 6 + h)^t$.