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FEM for BVP

Error estimates for cor(1)

Chapter VI: Numerical methods for IVP

<u>Goal</u>: Present basic numerical methods for

$$(\mathrm{IVP}) \quad \begin{cases} \dot{y}(t) = f(y(t)) & \text{for } t \in (0,T] \\ y(0) = y_0, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$, $y_0 \in \mathbb{R}$, $T > 0$ are given

y is unknown

1) Finite differences (FD):

Consider $y : \mathbb{R} \to \mathbb{R}$ differentiable in $t_0 \in \mathbb{R}$:

$$\dot{y}(t_0) = \lim_{h \to 0} \frac{y(t_0+h) - y(t_0)}{h}$$

This motivates:

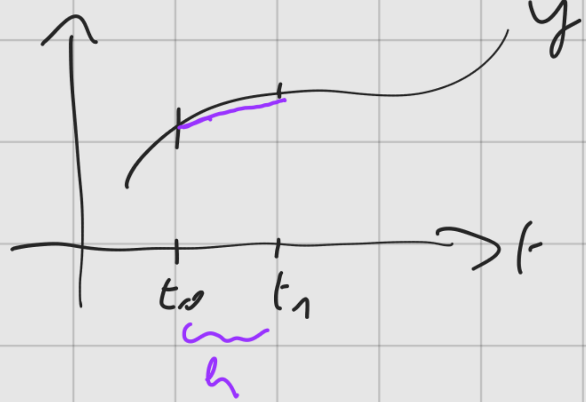
Def: For a fixed (small) $h > 0$ we define

Def: For a fixed (small) $h > 0$, we define the forward difference by

$$\dot{y}(t_0) \overset{\text{approx}}{\approx} \frac{y(t_0+h)-y(t_0)}{h} = \frac{y(t_1)-y(t_0)}{t_1-t_0},$$

where $t_1 := t_0 + h$.

Rem: The term $\frac{y(t_1)-y(t_0)}{t_1-t_0}$ is the slope of line connecting $(t_0, y(t_0))$ and $(t_1, y(t_1))$:



Similarly, one defines

backward difference $\dot{y}(t_0) \approx \frac{y(t_0)-y(t_0-h)}{h}$

centered difference $\dot{y}(t_0) \approx \frac{y(t_0+h)-y(t_0-h)}{2h}$

2) First time integrators for IVP

Recall: (IVP) $\begin{cases} \dot{y}(t) = f(y(t)) & \text{for } t \in (0,T] \\ y(0) = y_0 \end{cases}$

Idea: Euler (1768)

Consider $N \in \mathbb{N}$ (LARGE), define the time step $k := \frac{T}{N}$ and a time grid $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ with $t_{n+1} = t_n + k$.

For $t_0 \leq t \leq t_0 + k$, one approximates $\dot{y}(t)$ by forward difference:

$$f(y(t)) \underset{\text{IVP}}{=} \dot{y}(t) \approx \frac{y(t+k) - y(t)}{k} \qquad \text{or}$$

$$y(t+k) \approx y(t) + k \cdot f(y(t))$$

For $t = t_0$, one obtains:

$$\underbrace{y(t_0+k)}_{t_1} \approx \underbrace{y(t_0)}_{y_0} + k\cdot f(\underbrace{y(t_0)}_{y_0})$$

(grid) IVP

That is:

$$\underbrace{y(t_1)}_{\substack{\text{exact sol}\\ \text{(unknown)}}} \approx \underbrace{y_0 + k\cdot f(y_0)}_{\substack{\text{everything is}\\ \text{known!}}} = \underbrace{y_1}_{\substack{\text{numerical}\\ \text{approx.}}}$$

One can repeat this procedure and get

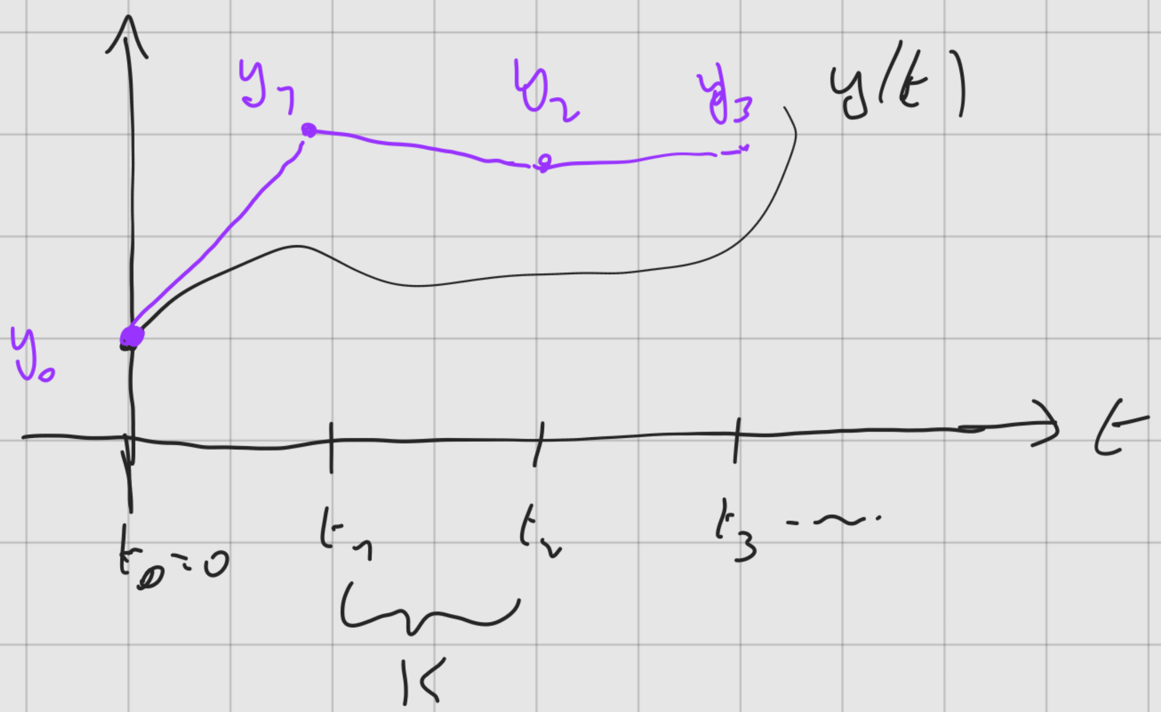
explicit Euler/forward Euler scheme:

$$\boxed{y_{n+1} = y_n + k\cdot f(y_n)} \qquad n = 0, 1, 2, \ldots$$

which gives approx. $y_n \approx y(t_n)$

at the time grid

Illustration:



Rem: If $k \to 0$, then error $\to 0$.

Similarly, one can define

implicit Euler / backward Euler scheme

$$y_{n+1} = y_n + k f(y_{n+1}) \quad \text{for } n = 0, 1, 2, \ldots$$

which gives approx. $y_n \approx y(t_n)$

Rem: Nonlinear system to solve

at each time step
(→ costly)

Similarly,

Crank-Nicolson scheme:

$$y_{n+1} = y_n + \frac{k}{2}\left(f(y_n) + f(y_{n+1})\right)$$

Rem: Can do more after PDE.

Rem:

	Exp Euler	Imp. Euler	CN
+	easy	stable (see PDE)	stable (see PDE)
−	too simple	implicit	implicit

Chapter VII: The heat equation

Goal: Briefly study exact solution
Present and study numerical approximations

FEM → space

implicit Euler → time

1) Heat equation in 1D:

Consider model problem:

Describe heat flow along thin wire/metal rod.

$$(H)\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & , 0<x<1,\ 0<t\leq T \quad \text{DE} \\ u(0,t)=0,\ u_x(1,t)=0 & , 0<t\leq T \quad \text{BC} \\ u(x,0)=u_0(x) & , 0<x<1 \quad \text{IC} \end{cases}$$

$u(x,t)$ → temperature in wire at position x and time t. (unknown)

$u_0(x)$ → initial temperature profile (given)

$f(x,t)$ → heat sources/sinks (given)

$T>0$ → final time (given)

$u(0,t)=0$ → hom. Dirichlet BC (fixed temp. 0 at $x=0$)

$u_x(1,t)=0$ → hom. Neumann BC (insulated BC, no heat flux at $x=1$)

Th: The sol. to heat eq. (H) satisfies

Thm: The sol. to heat eq. (H) satisfies the stability estimates:

(i) $$\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\, ds$$

(ii) $$\|u_x(\cdot,t)\|_{L^2(0,1)} \leq \|u_0'\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\, ds$$

(iii) When $f \equiv 0$, one has

$$\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} \cdot e^{-2t}$$

Rem: • Notation: $\|u(\cdot,t)\|_{L^2(0,1)} = \left(\int_0^1 |u(x,t)|^2 dx\right)^{1/2}$

• (iii) says that in "average" the temperature goes to zero.

Proof:

(i) Multiply (H) with u and integrate over $(0,1)$:

$$\underbrace{\int_0^1 u_t(x,t)\,u(x,t)\,dx}_{\frac{1}{2}\frac{d}{dt}\left(u^2(x,t)\right)} - \underbrace{\int_0^1 u_{xx}(x,t)\,u(x,t)\,dx}_{\text{by parts: } -u_x u\big|_0^1 + \int_0^1 u_x \cdot u_x\, dx} = \int_0^1 f(x,t)\,u(x,t)\,dx$$

$\frac{1}{2}\frac{d}{dt}$ $= 0$ due to BC

We obtain:

$$\frac{1}{2}\frac{d}{dt}\underbrace{\int_0^1 u^2(x,t)\,dx}_{\|u(\cdot,t)\|^2_{L^2(0,1)}} + \underbrace{\int_0^1 u_x^2(x,t)\,dx}_{\underbrace{\|u_x(\cdot,t)\|^2_{L^2(0,1)}}_{\geq 0}} = \int_0^1 f(x,t)\,u(x,t)\,dx \quad (*)$$

This gives us:

$$\underbrace{\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(0,1)}}_{\text{chain rule}} \leq \int_0^1 f(x,t)\cdot u(x,t)\,dx \overset{C\text{-}S}{\leq}$$

$$\frac{2}{2}\cdot\|u(\cdot,t)\|_{L^2(0,1)}\frac{d}{dt}\|u(\cdot,t)\|_{L^2(0,1)} \leq \|f(\cdot,t)\|_{L^2(0,1)}\cdot\|u(\cdot,t)\|_{L^2(0,1)}$$

Finally, we get

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^2(0,1)} \leq \|f(\cdot,t)\|_{L^2(0,1)}$$

that we integrate in time to get

$$\|u(\cdot,t)\|_{L^2(0,1)} - \|u_0\|_{L^2(0,1)} \leq \int^t \|f(\cdot,s)\|_{L^2(0,1)}\,ds$$

(ii) Similar, cf. book p. 179

(iii) Using (*) with $f \equiv 0$, one gets

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(0,1)} + \|u_x(\cdot,t)\|^2_{L^2(0,1)} = 0$$

Recall Poincaré: $\|u(\cdot,t)\|^2_{L^2(0,1)} \leq \frac{1}{2}\|u_x(\cdot,t)\|^2_{L^2(0,1)}$

or $2\|u(\cdot,t)\|^2_{L^2(0,1)} \leq \|u_x(\cdot,t)\|^2_{L^2(0,1)}$

This gives us

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(0,1)} + 2\|u(\cdot,t)\|^2_{L^2(0,1)} \leq 0$$

Multiply with integrating factor $2e^{4t}$ and get

$$\underbrace{\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(0,1)} e^{4t} + 4e^{4t}\|u(\cdot,t)\|^2_{L^2(0,1)}}_{\frac{d}{dt}\left(e^{4t}\cdot\|u(\cdot,t)\|^2_{L^2(0,1)}\right)} \leq 0$$

Finally, one integrates in time $\int_0$ and get

$$e^{4t}\|u(\cdot,t)\|^2_{L^2(0,1)} - \|u_0\|^2_{L^2(0,1)} \leq 0$$

That is $\|u(\cdot,t)\|^2_{L^2(0,1)} \leq e^{-4t}\|u_0\|^2_{L^2(0,1)}$:-)

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