David Cohen (david.cohen@chalmers.se) Chalmers & GU

Chapter 6: Numerical methods for IVP (summary)

February 4, 2022

Goal: Present basic numerical methods for the IVP

$$\begin{cases} \dot{y}(t) = f(y(t)) \quad \text{for} \quad t \in (0,T] \\ y(0) = y_0. \end{cases}$$

Here, $T > 0$, $y_0 \in \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ are given and $\dot{y}(t) = \frac{\mathrm{d}}{\mathrm{d}t} y(t)$. Observe that what is presented below can be adapted to the situation, where $f(t,y)$ and $y(t_0)$ or for vector-valued problems.

- For $y\colon \mathbb{R} \to \mathbb{R}$ differentiable at t_0 and a fixed $h > 0$, we define the following approximations of the derivative given by so-called finite differences:

  The forward difference
  $$\dot{y}(t_0) \approx \frac{y(t_0+h) - y(t_0)}{h}.$$
  The backward difference
  $$\dot{y}(t_0) \approx \frac{y(t_0) - y(t_0-h)}{h}.$$
  The centered difference
  $$\dot{y}(t_0) \approx \frac{y(t_0+h) - y(t_0-h)}{2h}.$$

- Consider the IVP
  $$\begin{cases} \dot{y}(t) = f(y(t)) \quad \text{for} \quad t \in (0,T] \\ y(0) = y_0. \end{cases}$$
  Let $N \in \mathbb{N}$ and define the time step $k = \frac{T}{N}$ as well as the time grid $0 = t_0 < t_1 < \ldots < t_N = T$, where $t_n = nk$ for $n = 0, 1, \ldots, N$.

  We define the following time integrators for the above IVP (starting with $y_0 = y(0)$):

  The (forward/explicit) Euler scheme
  $$y_{n+1} = y_n + kf(y_n).$$
  The backward/implicit Euler scheme
  $$y_{n+1} = y_n + kf(y_{n+1}).$$
  The Crank–Nicolson scheme
  $$y_{n+1} = y_n + \frac{k}{2}\big(f(y_n) + f(y_{n+1})\big).$$
  These provide numerical approximations $y_n \approx y(t_n)$ to the exact solution of the IVP on the time grid $(t_n)_{n=0}^N$.

Further resources:

- FD at www.wikipedia.org
- FD at brown.edu
- FD and Euler at ocw.mit.edu
- Euler at math.lamar.edu
- Euler at calcworkshop.com
- Euler at intmath.com