Proof of a posteriori error estimates

February 4, 2022

Notation. The BVP reads $-(a(x)u(x)')' = f(x)$ on $(0,1)$ with homogeneous Dirichlet BC.
It's $cG(1)$ approximation is denoted by u_h.

Proof. Set $e := u - u_h$ for the error. By the definition of the energy norm, one has

$$
\begin{aligned}
\|e\|_E^2 &= \int_0^1 a(x)e'(x)\,(e(x)-\pi_h e(x))'\,\mathrm{d}x \\
&= \sum_{j=1}^{m+1} \int_{x_{j-1}}^{x_j} a(x)e'(x)\,(e(x)-\pi_h e(x))'\,\mathrm{d}x \\
&= \sum_{j=1}^{m+1} \left(-\int_{x_{j-1}}^{x_j} (a(x)e'(x))'(e(x)-\pi_h e(x))\,\mathrm{d}x + \left(a(x)e'(x)(e(x)-\pi_h e(x))\right)\Big|_{x=x_{j-1}}^{x_j} \right) \\
&= -\sum_{j=1}^{m+1} \int_{x_{j-1}}^{x_j} (a(x)e'(x))'(e(x)-\pi_h e(x))\,\mathrm{d}x,
\end{aligned}
$$

where one has used Galerkin orthogonality, the definition of the partition, integration by parts and the fact that the interpolant $\pi_h e$ and e are equals on the grid $\{x_j\}_j$.

Next, we examine the term $-(a(x)e'(x))'$ and get the relation

$$
-(a(x)e'(x))' = -(a(x)(u(x)-u_h(x))')' = f(x) + (a(x)u_h'(x))' = R(u_h(x)),
$$

where we have used the definition of e, the fact that u solves our BVP, and the definition of the residual $R(u_h)$.

Inserting this relation in the above, one then obtains

$$
\begin{aligned}
\|e\|_E^2 &= \int_0^1 R(u_h(x))\,(e(x)-\pi_h e(x))\,\mathrm{d}x \\
&= \int_0^1 \frac{h(x)}{\sqrt{a(x)}} R(u_h(x)) \frac{\sqrt{a(x)}}{h(x)}\,(e(x)-\pi_h e(x))\,\mathrm{d}x \\
&\leq \left(\int_0^1 \frac{h(x)^2}{a(x)} R^2(u_h(x))\,\mathrm{d}x \right)^{1/2} \left(\int_0^1 a(x) \left(\frac{e(x)-\pi_h e(x)}{h(x)} \right)^2 \mathrm{d}x \right)^{1/2} \\
&\leq \left(\int_0^1 \frac{h(x)^2}{a(x)} R^2(u_h(x))\,\mathrm{d}x \right)^{1/2} \left\| \frac{e-\pi_h e}{h} \right\|_{L_a^2} \\
&\leq C \left(\int_0^1 \frac{h(x)^2}{a(x)} R^2(u_h(x))\,\mathrm{d}x \right)^{1/2} \|e'\|_{L_a^2} \\
&\leq C \left(\int_0^1 \frac{h(x)^2}{a(x)} R^2(u_h(x))\,\mathrm{d}x \right)^{1/2} \|e\|_E,
\end{aligned}
$$

where one uses Cauchy–Schwartz and an interpolant estimates, see Remark 5.3 on page 131 of the book, and the definition of the energy norm.

One concludes the proof by simplifying one energy norm and get

$$
\|e\|_E \leq C \left(\int_0^1 \frac{h(x)^2}{a(x)} R^2(u_h(x))\,\mathrm{d}x \right)^{1/2}.
$$