R

H

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) \\ u(0,t) = 0, \quad u_x(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

FEM $\leftrightarrow x$

Euler $\leftrightarrow t$

Estimates (i) $\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\, ds$

(iii) $f \equiv 0$: $\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} e^{-2t}$

Use: $\frac{1}{2}\frac{d}{ds}\|u(\cdot,s)\|_{L^2}^2 + \|u_x(\cdot,s)\|_{L^2}^2 = 0$

- Quiz, feedback, exam

Application of (i): Continuous dependence of sol. to (H) on data:

If u_1, resp. u_2 are sol. to (H) with RHS (= right hand side) f_1, resp. f_2 and IC v_1 and v_2: Then, using (i) above, one gets

[$u_1 - u_2$ solves PDE $u_t - u_{xx} = f_1 - f_2$, $u(x,0) = v_1(x) - v_2(x)$]

$$\|u_1(\cdot,t) - u_2(\cdot,t)\|_{L^2} \leq \|v_1 - v_2\|_{L^2} + \int_0^t \|f_1(\cdot,s) - f_2(\cdot,s)\|_{L^2}\, ds$$

We next show another technical result

<u>Th:</u> ("energy estimate")
Consider (H) with $f \equiv 0$ and let $\varepsilon > 0$. Then, one has the estimate

$$\int_\varepsilon^t \|u_t(\cdot,s)\|_{L^2(0,1)}\,ds \leq \frac{1}{2}\sqrt{\ln\left(\frac{t}{\varepsilon}\right)}\cdot\|u_0\|_{L^2(0,1)} \quad \forall t\in(0,T].$$

<u>Proof:</u>

- First, multiply PDE with $-t\,u_{xx}$, integrate by parts $\int_0^1 \dots dx$ to get:

$$\underbrace{-t\int_0^1 u_t(x,t)\,u_{xx}(x,t)\,dx}_{\text{by parts}} + t\underbrace{\int_0^1 (u_{xx}(x,t))^2\,dx}_{\|u_{xx}(\cdot,t)\|^2_{L^2} \text{ (Def norm)}} = 0$$

$$\underbrace{u_t(x,t)u_x(x,t)\Big|_{x=0}^{1}}_{=0} - \int_0^1 \underbrace{u_{tx}(x,t)\,u_{xx}}_{\frac{1}{2}\frac{d}{dt}(u_x^2)}(x,t)\,dx$$

This gives us: $\frac{t}{2}\frac{d}{dt}\|u_x(\cdot,t)\|^2_{L^2} + t\|u_{xx}(\cdot,t)\|^2_{L^2} = 0$

- Next, one observes that:

$$\frac{d}{dt}\left(t \cdot \|u_x(\cdot,t)\|_{L^2}^2\right) \overset{\text{product rule}}{=} \|u_x(\cdot,t)\|_{L^2}^2 + t\,\frac{d}{dt}\|u_x(\cdot,t)\|_{L^2}^2$$

that is $\quad t\,\frac{d}{dt}\|u_x(\cdot,t)\|_{L^2}^2 = \frac{d}{dt}\left(t\cdot\|u_x(\cdot,t)\|_{L^2}^2\right) - \|u_x(\cdot,t)\|_{L^2}^2$

All in all, we obtain the relation

$$\frac{d}{dt}\left(t\cdot\|u_x(\cdot,t)\|_{L^2}^2\right) + 2t\,\|u_{xx}(\cdot,t)\|_{L^2}^2 = \|u_x(\cdot,t)\|_{L^2}^2$$

- We integrate the above over time $\int_0^t \dots ds$!

$$\int_0^t \frac{d}{ds}\left(s\cdot\|u_x(\cdot,s)\|_{L^2}^2\right) ds + 2\cdot\int_0^t s\,\|u_{xx}(\cdot,s)\|_{L^2}^2\, ds =$$

$$= \int_0^t \|u_x(\cdot,s)\|_{L^2}^2\, ds \underset{\boxed{\times}}{\leq} \frac{1}{2}\|u_0\|_{L^2}^2$$

$\boxed{\times}$: We recall the relation

$$\frac{1}{2}\frac{d}{ds}\|u(\cdot,s)\|_{L^2}^2 + \|u_x(\cdot,s)\|_{L^2}^2 = 0$$

from the proof of the previous theorem.

Integrate with respect to time gives:

$$\int_0 \left\{ \frac{1}{2} \frac{d}{ds} \|u(\cdot,s)\|_{L^2}^2 + \|u_x(\cdot,s)\|_{L^2}^2 \right\} ds = 0 \iff$$

$$\underbrace{\frac{1}{2}\|u(\cdot,t)\|_{L^2}^2}_{(\text{"}s=t\text{"})} - \underbrace{\frac{1}{2}\|u_0\|_{L^2}^2}_{(\text{"}s=0\text{"})} + \int_0^t \|u_x(\cdot,s)\|_{L^2}^2 ds = 0$$

This gives us the bound

$$\int_0^t \|u_x(\cdot,s)\|_{L^2}^2 ds = \frac{1}{2}\|u_0\|_{L^2}^2 - \underbrace{\frac{1}{2}\|u(\cdot,t)\|_{L^2}^2}_{\geq 0} \leq \frac{1}{2}\|u_0\|_{L^2}^2$$

- The above thus provides:

$$\underbrace{\int_0^t \frac{d}{ds}\left(s\|u_x(\cdot,s)\|_{L^2}^2\right) ds}_{\substack{t\|u_x(\cdot,t)\|_{L^2}^2 - 0 \\ (\text{"}s=t\text{"}) \quad (\text{"}s=0\text{"})}} + 2\int_0^t s\|u_{xx}(\cdot,s)\|_{L^2}^2 ds \leq \frac{1}{2}\|u_0\|_{L^2}^2$$

$$\Rightarrow \quad t\|u_x(\cdot,t)\|_{L^2}^2 + 2\int_0^t s\|u_{xx}(\cdot,s)\|_{L^2}^2 ds \leq \frac{1}{2}\|u_0\|_{L^2}^2$$

Hence (i) $t\|u_x(\cdot,t)\|_{L^2}^2 \leq \frac{1}{2}\|u_0\|_{L^2}^2$

(ii) $2\int_0^t s\|u_{xx}(\cdot,s)\|_{L^2}^2 ds \leq \frac{1}{2}\|u_0\|_{L^2}^2$

- Finally, we go back to what we were

interested in: $u_t - u_{xx} = 0$ or $u_t = u_{xx}$:

$$\int_\varepsilon^t \|u_t(\cdot,s)\|_{L^2}\, ds \overset{u_t = u_{xx}}{=} \int_\varepsilon^t \|u_{xx}(\cdot,s)\|_{L^2}\, ds =$$

$$= \int_\varepsilon^t \frac{1}{\sqrt{s}} \cdot \sqrt{s}\, \|u_{xx}(\cdot,s)\|_{L^2}\, ds \overset{C\text{-}S}{\leq}$$

$$\left(\int_\varepsilon^t \left(\frac{1}{\sqrt{s}}\right)^2 ds \right)^{1/2} \cdot \left(\int_\varepsilon^t s \|u_{xx}(\cdot,s)\|_{L^2}^2\, ds \right)^{1/2} \leq$$

$$\underset{(ii)}{\leq} \left(\ln(t) - \ln(\varepsilon) \right)^{1/2} \cdot \left(\frac{1}{4} \|u_0\|_{L^2}^2 \right)^{1/2}$$

$$\leq \frac{1}{2} \sqrt{\ln\left(\frac{t}{\varepsilon}\right)} \cdot \|u_0\|_{L^2(0,1)}$$

:-)

2) Discretisation of the heat equation:

Consider the following PDE

$$(H) \begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0 < x < 1,\ 0 < t \leq T \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

(i) To find a variational formulation (in space), consider trial/test space :

$H_0^1(0,1) = \{ v:[0,1] \to \mathbb{R},\ v, v' \in L^2(0,1) \text{ and } v(0) = v(1) = 0 \}$

As for BVP, we multiply PDE with test fct $v \in H_0^1$, integrate by parts and get:

$$\int_0^1 u_t(x,t)\cdot v(x)\,dx - \underbrace{\int_0^1 u_{xx}(x,t)\, v(x)\,dx}_{\underbrace{u_x(x,t)v(x)\Big|_{x=0}^{1}}_{=0 \text{ since } v \in H_0^1} - \int_0^1 u_x(x,t)\, v'(x)\,dx} = \int_0^1 f(x,t)\, v(x)\,dx$$

That is $\displaystyle\int_0^1 u_t(x,t)\cdot v(x)\,dx + \int_0^1 u_x(x,t)\, v'(x)\,dx = \int_0^1 f(x,t)\, v(x)\,dx$

We thus obtain the VF:

(VF) For each $0 < t \le T$, find $u(\cdot,t) \in H_0^1$ s.t.

$$(u_t(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v')_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in H_0^1$$

$$u(x,0) = u_0(x)$$

(ii) To get a FE prob., we consider a uniform partition of $[0,1]$; $T_h: x_0 = 0 < x_1 < x_2 < \dots < x_{m+1} = 1$, with $x_j - x_{j-1} = h = \frac{1}{m+1}$.

Consider space

$$V_h^0 = \{ v:[0,1] \to \mathbb{R} : v \text{ cont, pw linear on } T_h \text{ and } v(0) = v(1) = 0 \}$$

$$= \operatorname{span}(\varphi_1, \varphi_2, \dots, \varphi_m), \text{ where}$$

φ_j are hat functions.

Obs: $V_h^0 \subset H_0^1$.

The FE then reads

(FE) For each $0 < t \le T$, find $u_h(\cdot, t) \in V_h^0$ s.t.

$$(u_{h,t}(\cdot,t), v_h)_{L^2} + (u_{h,x}(\cdot,t), v_h')_{L^2} = (f(\cdot,t), v_h)_{L^2}$$

$$u_h(x,0) = \Pi_h u_0(x) \qquad \forall v_h \in V_h^0.$$

cont pw. linear interpolant of u_0.

(iii) We now obtain a system of linear ODE by writing $u_h(x,t) = \sum_{j=1}^{m} \xi_j(t) \cdot \varphi_j(x)$ and $v_h = \varphi_i$:

Insert into (FE) to get:

$$\sum_{j=1}^{m} \dot{\xi}_j(t) \underbrace{(\varphi_j, \varphi_i)_{L^2}}_{m_{ij}} + \sum_{j=1}^{m} \xi_j(t) \overbrace{(\varphi_j', \varphi_i')_{L^2}}^{s_{ij}} = \overbrace{(f(\cdot,t), \varphi_i)_{L^2}}^{F_i(t)}$$

for $i = 1, 2, \dots, m$

This gives us the syst. of ODE:

$$M \dot{\xi}(t) + S \xi(t) = F(t)$$

$$\xi(0),$$

where

$M = (m_{ij})^{m} \rightarrow$ mass matrix

$M = (m_{ij})_{i,j=1}^{m} \to$ mass matrix (see above)

$S = (s_{ij})_{i,j=1}^{m} \to$ stiffness matrix (see above)

$F(t) = (F_i(t))_{i=1}^{m} \to$ "load vector"

$$\xi(t) = \begin{pmatrix} \xi_1(t) \\ \vdots \\ \xi_m(t) \end{pmatrix} \to \text{unknown}$$

$$\xi(0) = \begin{pmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_m) \end{pmatrix}$$ (def interpolant)

(iv) Finally, one uses f. ex. implicit Euler scheme to approximate $\xi(t)$ at time grid points $0 = t_0 < t_1 < \ldots < t_N = T$, $t_j - t_{j-1} = k = \frac{T}{N}$.

$$M\left(\frac{\underline{\xi^{(n+1)}} - \xi^{(n)}}{k}\right) + S\,\underline{\xi^{(n+1)}} = F(t_{n+1})$$

$$\Leftrightarrow \left(\underline{M} + k\,\underline{S}\right) \cdot \xi^{(n+1)} = M\,\xi^{(n)} + k\,F(t_{n+1}) \quad \text{for } n = 0,1,2,\ldots$$

$(Ax = b)$

$$\xi^{(0)} = \xi(0)$$

Computing $\xi^{(n)}$ gives approx. of $\xi(t_n)$ and thus $u_h^n(x, t_n) \approx u(x, t_n)$ of heat eq. (H)

Rem: • OK to use C-N, but for exp. Euler one needs to be carefull (see last ex. in Ass. 1)

• For each time step → solve linear syst. $(Ax=b)$

• If F is complicated → use quadrature rule.