R

Wave eq. (W) $\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t) \\ u(0,t) = 0,\ u(1,t) = 0 \\ u(x,0) = u_0,\ u_t(x,0) = v_0 \end{cases}$

↓

VF: For each $0<t\leq T$, find $u(\cdot,t) \in H_0^1(0,1)$ s.t.

$$(u_{tt}(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v')_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in H_0^1$$

+ IC

↓

FE problem for $u_h(x,t) = \sum_{j=1}^{m} \underbrace{\xi_j(t)}_{\text{unkn.}} \underbrace{\varphi_j(x)}_{\text{hat}}$

↓

Syst. of ODE

$M \ddot{\xi}(t) + S\, \xi(t) = F(t)$

+ IC

$$\left( S = \frac{1}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \qquad M = \frac{h}{6} \begin{pmatrix} 4 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 4 \end{pmatrix} \right)$$

↓

CN with step size $k = \frac{T}{N}$ gives approx $\xi^{(n)} \approx \xi(t_n)$

$t_n = n \cdot k$

Quiz today → Sat. 13:00

Proof (Conservation discrete energy by C-N)

We obtain: $(\xi^{(n+1)} + \xi^{(n)})^T S (\xi^{(n+1)} - \xi^{(n)}) +$

$(\xi^{(n+1)} - \xi^{(n)})^T M (\xi^{(n+1)} - \xi^{(n)}) + \ldots$

$+ (\eta \;\; + \eta \;\;) M (\eta \;\; - \eta \;\;) + 0 = 0$

$(f \equiv 0 \text{ in } (\omega))$

2 terms disappear for C-N :-)

The above gives us the relation:

since $S'$ symmetric

$$\zeta^{(n+1)^T} S' \zeta^{(n+1)} - \cancel{\zeta^{(n+1)^T} S' \zeta^{(n)}} + \cancel{\zeta^{(n)^T} S' \zeta^{(n+1)}} - \zeta^{(n)^T} S' \zeta^{(n)} + \eta^{(n+1)^T} M \eta^{(n+1)} - \cancel{\eta^{(n+1)^T} M \eta^{(n)}} + \cancel{\eta^{(n)^T} M \eta^{(n+1)}} - \eta^{(n)^T} M \eta^{(n)} = 0$$

since $M$ is symmetric

That is

$$\underbrace{\underset{(*)}{\zeta^{(n+1)^T} S' \zeta^{(n+1)}} + \underset{(**)}{\eta^{(n+1)^T} M \eta^{(n+1)}}}_{\text{discrete energy at time } t_{n+1}} = \underbrace{\zeta^{(n)^T} S' \zeta^{(n)} + \eta^{(n)^T} M \eta^{(n)}}_{\text{discrete energy at time } t_n}$$

$\leadsto$ discrete energy conserved :-)

Obs: $(*)$ is in fact $\| u^{n+1}_{h,x}(\cdot, t_{n+1}) \|^2_{L^2(0,1)}$

and $(**)$ is in fact $\| u^{n+1}_{h,t}(\cdot, t_{n+1}) \|^2_{L^2(0,1)}$

$\boxtimes$

Rem: • What is the purpose of conservation of energy?

(i) exact sol. has this property

$\Rightarrow$ Want numerical sol. to have the same property.

(ii) this tells us that one has a kind of stability for the numerical sol..

- What can we say if (W) has an $f$?

If $f$ comes from a potential, then one can derive advanced numerical methods that preserves the energy of the problem.

## Chapter IX: On our way to FEM in 2d

Goal: Extend integration by parts in 2d, extend VF in 2d, pw linear interpolation, FEM in 2d (OK for 3d)

### 1) Green's formula:

Integration by parts in 1d: $\int u'v = uv| - \int uv'$ is extended in 2d thanks to

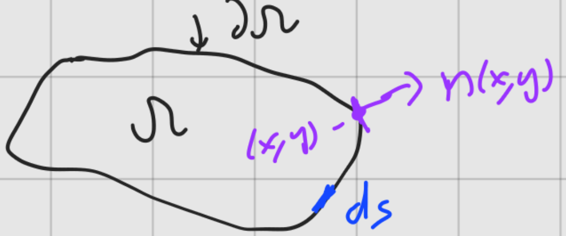
Th: (Green's formula / Green's first identity)

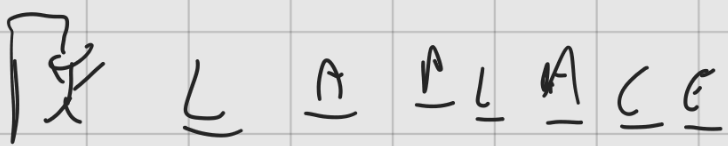
Let $\Omega \subset \mathbb{R}^2$ open bounded domain with it's boundary $\partial\Omega$ pw smooth. For $u \in C^2(\Omega)$ and $v \in C(\Omega)$, one has

$$\iint_\Omega \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) v \, dx\,dy = \int_{\partial\Omega} \left(\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot n\right) v \, ds$$

$$- \iint_\Omega \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) dx\,dy,$$

where $u = u(x,y)$, $v = v(x,y)$, $n = n(x,y)$ is the outward unit normal at the boundary at point $(x,y) \in \partial\Omega$ and $ds$ is a curve element on $\partial\Omega$.



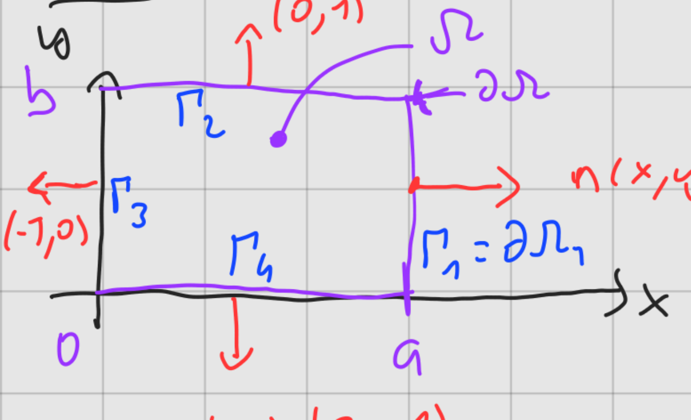




<u>Short</u>: $\int_\Omega \Delta u \, v \, dx\,dy = \int_{\partial\Omega} (\nabla u \cdot n) v \, ds - \int_\Omega \nabla u \cdot \nabla v \, dx\,dy$

<u>Notation</u>: $\frac{\partial u}{\partial n} = n \cdot \nabla u$

(Proof: Divergence)

<u>Proof</u>: Only for rectangle $\Omega = (0,a) \times (0,b)$



$n(x,y)=(0,-1)$

- We start with

$$\iint_\Omega \frac{\partial^2 u}{\partial x^2}(x,y)\,v(x,y)\,dx\,dy \overset{\text{Def }\Omega}{=} \int_0^b \int_0^a \frac{\partial^2 u}{\partial x^2}(x,y)\,v(x,y)\,dx\,dy \overset{\text{by parts in 1d}}{=}$$

$$= \int_0^b \left( \frac{\partial u}{\partial x}(x,y)\,v(x,y)\Big|_{x=0}^{a} - \int_0^a \frac{\partial u}{\partial x}(x,y)\,\frac{\partial v}{\partial x}(x,y)\,dx \right) dy =$$

$$= \int_0^b \left( \underline{\frac{\partial u}{\partial x}(a,y)\,v(a,y)} - \underline{\frac{\partial u}{\partial x}(0,y)\,v(0,y)} \right) dy - \underbrace{\iint_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx\,dy}_{\text{Def }\Omega} \quad (*)$$

- Next, we observe that $n(x,y)=(1,0)$ on $\Gamma_1=\partial\Omega_1$
This is used to rewrite the term

$$\int_0^b \underline{\frac{\partial u}{\partial x}(a,y)\,v(a,y)}\,dy = \int_0^b \left( \frac{\partial u}{\partial x}\underbrace{(a,y)}_{(x,y)}, \frac{\partial u}{\partial y}\overbrace{(a,y)}^{(x,y)} \right)\cdot \underbrace{(1,0)}_{\substack{\| \\ n(x,y) \text{ on } \Gamma_1}} v\overbrace{(a,y)}^{(x,y)}\,dy =$$

$(x=a$ on $\Gamma_1)$

$$= \int_{\Gamma_1} \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot n\, v\, ds$$

We can do the same for the term in red

above observing that on $\Gamma_3 = \Omega_3$, one has $n(x,y) \equiv (-1,0)$:

$$\int_0^b \left(-\frac{\partial u}{\partial x}(0,y)\, v(0,y)\right) dy = \int_{\Gamma_3} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot n \, v \, ds$$

$\Rightarrow$ (*) reads now:

$$\iint_\Omega \frac{\partial^2 u}{\partial x^2}(x,y)\, v(x,y)\, dx\, dy = \int_{\Gamma_1 \cup \Gamma_3} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot n \, v \, ds$$

$$- \iint_\Omega \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}\, dx\, dy$$

- The same can be done for the term

$$\iint_\Omega \frac{\partial^2 u}{\partial y^2}(x,y)\, v(x,y)\, dx\, dy \quad \text{using } \Gamma_2 \cup \Gamma_4.$$

$\Rightarrow$ Adding these 2 formulas gives us Green's formula

## 2) Variational formulation in 2d:

(Consider ...

Consider Poisson's eq. on a (nice) domain $\Omega \subset \mathbb{R}^2$:

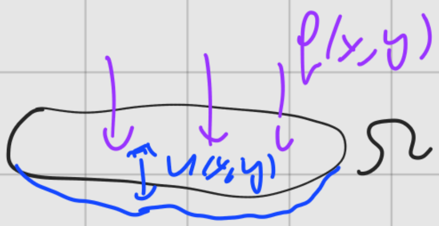
$$(P) \begin{cases} -\Delta u(x,y) = f(x,y) & \text{in } \Omega \\ u(x,y) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f: \Omega \to \mathbb{R}$ (nice) given, and

$$\Delta u(x,y) = \nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Applications:

- Simple model for displacement of a membrane (f.ex. drum skin)




- Newtonian gravitation (describe gravitational field)
- Electrostatics (describe static electricity objects)
- ... etc.

To find a VF for (P) we proceed as

To find a VF for (P), one proceeds as in 1d: test with $v$ that vanishes on $\partial\Omega$, integrate, go Green:

GREEN

$$\iint_\Omega f(x,y)v(x,y)\,dxdy = -\iint_\Omega \Delta u(x,y)v(x,y)\,dxdy \overset{\text{GREEN}}{=}$$

$$= \underbrace{-\int_{\partial\Omega} (\nabla u \cdot n) v\,ds}_{=0 \text{ since } v\equiv 0 \text{ on } \partial\Omega} + \iint_\Omega \nabla u \cdot \nabla v\,dxdy$$

Next, consider the spaces

$$H^1(\Omega) = \{v : \Omega \to \mathbb{R} : \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} < \infty\}$$

$$H^1_0(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$$

$\hookrightarrow$ We get the VF for (P):

(VF) Find $u \in H^1_0(\Omega)$ s.t. $(\nabla u, \nabla v)_{L^2} = (f, v)_{L^2}$ $\forall v \in H^1_0$

__Rem__: • Using C-S, one gets

$$|(f,v)_{L^2}| \le \|f\|_{L^2} \cdot \|v\|_{L^2} < \infty$$

$f$ nice $\hookleftarrow$ $\quad$ $\hookrightarrow v \in H^1_0$

The integrals in (VF) are finite :-)

- Equivalence (VF) $\Leftrightarrow$ strong (F) depends on regularity of $f$ and shape of $\Omega$.

## 3) Triangulation:

Let $\Omega \subset \mathbb{R}^2$ be bounded domain with polygonal boundary $\partial\Omega$

Rem: Also OK if $\partial\Omega$ is smooth

Def: • A triangulation, or mesh, $T_h$ of $\Omega$ is a set $\{K\}$ triangles $K$ s.t.

$$\Omega = \bigcup_{K \in T_h} K \quad \text{and s.t.}$$

$$K_1 \cap K_2 = \begin{cases} \text{edge} \\ \text{or} \\ \text{corner} \\ \text{or} \\ \text{empty} \end{cases} \quad \forall\, K_1, K_2 \in T_h.$$

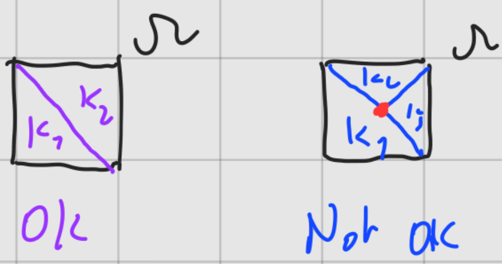
- The corner of triangle is called a node

- The local mesh size of a triangle $K$ is
  $h_K$: length of longest edge of $K$

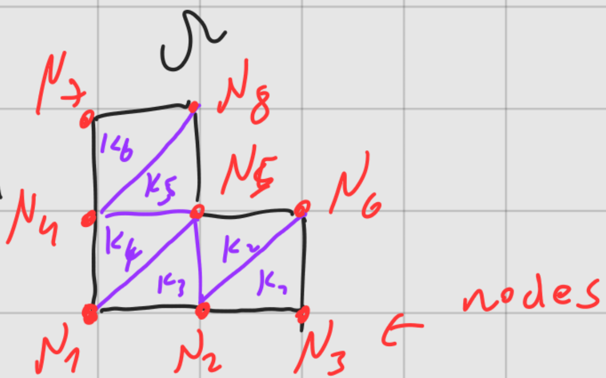
- The global mesh size is $h = \max_{K \in \mathcal{T}_h} h_K$

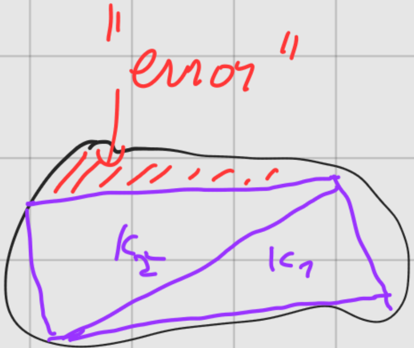
Rem:

- No triangle corners are hanging!
  I.e. lies on an edge of another triangle




- L-shaped domain

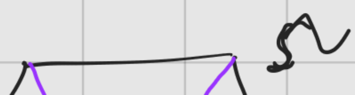


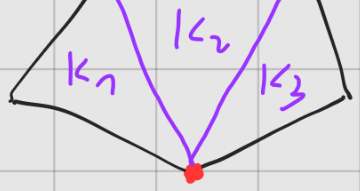

- 


Ω smooth

(no concern for the moment)

- If Ω is polygonal domain → Euler →
  ∃ triangulation: fan triangulation






select corner → link to other corners

- Difficult to generate mesh in 2d/3d.

  ∃ mesh generator:

  2d Delaunay mesh generator
  (Matlab pdetool)

- All triangles are nice in this lecture

not ok (here)