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- Piazza?
- Synchro
- Red path:

IVP $\dot{y} = f(y)$ + IC
numerical methods
Euler / C-N

BVP $-u''(x) = f(x)$
+ BC
numerical meth.
cG(1) FEM
$\Downarrow$
error estimate

Heat eq. $u_t(x,t) - u_{xx}(x,t) = f(x,t)$
+ BC + IC

In general no explicit formula for exact sol.

$\Downarrow$

numerical methods
((linear) PDE)

$\downarrow$ VF to PDE
$\downarrow$ FE problem (cG(1), space)
$\downarrow$ Euler / C-N (time)
$\downarrow$ u_h^n numerical approx.

Chapter VIII: The wave equation

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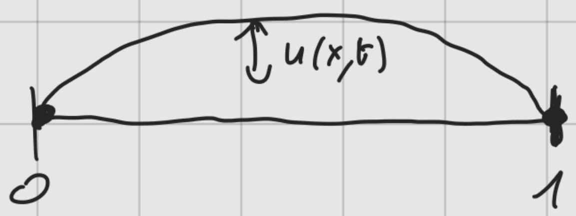
Goal: Briefly study exact sol.
Present a numerical discretisation

1) The wave equation in 1d:

Model prob: Describe vibration of string
(violin, guitar)



$$(W)\quad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t) & \text{PDE} \\ u(0,t) = 0,\ u(1,t) = 0 & \text{BC} \\ u(x,0) = \underbrace{u_0(x)}_{\text{position}},\ u_t(x,0) = \underbrace{v_0(x)}_{\text{velocity}}, & \text{IC} \end{cases}$$

where f, u_0, v_0 (nice) given,
Here, $0 \le x \le 1$, $0 \le t \le T$.

Th: Consider the homogeneous wave eq,
(that (W) with $f \equiv 0$), then one has
conservation of energy:

$$\underbrace{\frac{1}{2}\|u_t(\cdot,t)\|^2_{L^2(0,1)}}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\|u_x(\cdot,t)\|^2_{L^2(0,1)}}_{\text{potential energy}} =$$

total energy

$$= \underbrace{\frac{1}{2}\|v_0\|^2_{L^2(0,1)}}_{\text{initial kinetic energy}} + \underbrace{\frac{1}{2}\|u_0'\|^2_{L^2(0,1)}}_{\text{initial pot. energy}} = \text{Const} \quad \forall\, 0 \le t \le T$$

initial total energy.

Proof:

- Idea: test PDE with u_t, integrate by parts:

$$\int_0^1 \underbrace{u_{tt}(x,t)\,u_t(x,t)}_{\frac{1}{2}\frac{d}{dt}(u_t^2)}\,dx - \underbrace{\int_0^1 u_{xx}(x,t)\,u_t(x,t)\,dx}_{\underbrace{u_x u_t \Big|_{x=0}^{1}}_{=0 \text{ Dirichlet BC}} - \int_0^1 \underbrace{u_x(x,t)\,u_{tx}(x,t)}_{\frac{1}{2}\frac{d}{dt}(u_x^2)}\,dx} = 0$$

That is, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left(\underbrace{\int_0^1 u_t(x,t)^2\,dx}_{\|u_t(\cdot,t)\|^2_{L^2}} + \underbrace{\int_0^1 u_x(x,t)^2\,dx}_{\|u_x(\cdot,t)\|^2_{L^2}} \right) = 0 \quad \text{(Def norm)}$$

- Integrating in time $\int_0^t$ gives us:

$$\underbrace{\frac{1}{2}\|u_t(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|u_x(\cdot,t)\|_{L^2}^2}_{(\text{time } t)} = \underbrace{\frac{1}{2}\|v_0\|_{L^2}^2 + \frac{1}{2}\|u_0'\|_{L^2}^2}_{(\text{time } 0)} \quad :-)$$

By introducing a new variable $v = u_t$ for the velocity, we can rewrite (W) thus a system of first order (in time) PDE:

$$\begin{cases} u_t = v \\ v_t \overset{\text{Def } v}{=} u_{tt} \overset{(W)}{=} + u_{xx} + f \end{cases} \iff \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}_t}_{W_t} = \underbrace{\begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{W} + \underbrace{\begin{pmatrix} 0 \\ f \end{pmatrix}}_{F}$$

$$\hookrightarrow \quad W_t = A W + F$$

Rem: • The above representation (W) allows to apply C-N, Euler, ... (1st order)

• The above representation (W) allows to derive a $\underset{\text{space}}{c(h^1)} - \underset{\text{time}}{c(h^1)}$ discretisation (Book p. 194 if you want)

2) Discretisation of wave eq. in 1d:

We apply FEM in space first, and then C-N in time.

(i) To find a VF for wave eq. (W), consider space $H_0^1(0,1)$ and test with $v \in H_0^1(0,1)$ and integrate by parts:

$$\int_0^1 u_{tt}(x,t)v(x)dx - \underbrace{\int_0^1 u_{xx}(x,t)v(x)dx}_{\underbrace{u_x v\Big|_{x=0}^{1}}_{=0 \text{ since } v\in H_0^1} - \int_0^1 u_x(x,t)v_x(x)dx} = \int_0^1 f(x,t)v(x)dx$$

This gives the VF:

(VF) For each $0<t\leq T$, find $u(\cdot,t)\in H_0^1$ s.t.

$$(u_{tt}(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in H_0^1$$

$$u(x,0) = u_0(x)$$

$$u_t(x,0) = v_0(x)$$

(ii) For the FE prob., one considers

$$V_h^0 = \operatorname{span}(\varphi_1, \varphi_2, \dots, \varphi_m) \subset H_0^1$$

hat function

and obtain

(FE) For each $0 < t \leq T$, find $u_h(\cdot, t) \in V_h^0$ s.t.

$$(u_{h,tt}(\cdot,t), v_h)_{L^2} + (u_{h,x}(\cdot,t), v_{h,x})_{L^2} = (f(\cdot,t), v_h)_{L^2}$$

$$u_h(x,0) = \Pi_h u_0(x)$$

$$u_{h,t}(x,0) = \Pi_h v_0(x) \qquad \forall v_h \in V_h^0$$

$\hookrightarrow$ pw linear interpolant of u_0, resp. v_0.

(iii) To get a syst. of ODE from (FE), we write

$$u_h(x,t) = \sum_{j=1}^{m} \underbrace{\xi_j(t)}_{\text{unknown}} \varphi_j(x)$$ and take $v_h = \varphi_i$, $i = 1, 2, \dots, m$

This gives:

$$\sum_{j=1}^{m} \ddot{\xi}_j(t) \underbrace{(\varphi_j, \varphi_i)_{L^2}}_{\text{mass matrix } M} + \sum_{j=1}^{m} \xi_j(t) \underbrace{(\varphi_j', \varphi_i')_{L^2}}_{\text{stiffness matrix } S} = \underbrace{(f(\cdot,t), \varphi_i)}_{F(t)} \quad \text{for } i = 1, \dots, m$$

$\hookrightarrow$ ODE:

$$\begin{cases} M\ddot{\xi}(t) + S'\xi(t) = F(t) \\ \xi(0) = \begin{pmatrix} u_0(x_1) \\ \vdots \\ u_0(x_m) \end{pmatrix}, \quad \dot{\xi}(0) = \begin{pmatrix} v_0(x_1) \\ \vdots \\ v_0(x_m) \end{pmatrix} \end{cases}$$

(iv) In order to apply C-N, we rewrite above ODE as a first order system (see exact sol.)

$$\begin{cases} M\dot{\xi} = M\eta \\ M\dot{\eta} = -S'\xi(t) + F(t) \end{cases}$$

An application of C-N with step size

$K = \frac{T}{N}$ gives:

$$\begin{cases} M\left(\dfrac{\xi^{(n+1)} - \xi^{(n)}}{K}\right) = M\left(\dfrac{\eta^{(n+1)} + \eta^{(n)}}{2}\right) \\ M\left(\dfrac{\eta^{(n+1)} - \eta^{(n)}}{K}\right) = -S'\left(\dfrac{\xi^{(n+1)} + \xi^{(n)}}{2}\right) + \dfrac{F(t_{n+1}) + F(t_n)}{2} \end{cases}$$

This gives the recursion:

$$\begin{pmatrix} M & -\frac{k}{2}M \\ \frac{k}{2}S & M \end{pmatrix} \begin{pmatrix} \zeta^{(n+1)} \\ \eta^{(n+1)} \end{pmatrix} = \begin{pmatrix} M & \frac{k}{2}M \\ -\frac{k}{2}S & M \end{pmatrix} \begin{pmatrix} \zeta^{(n)} \\ \eta^{(n)} \end{pmatrix} + k \cdot \begin{pmatrix} 0 \\ \frac{F(t_{n+1})+F(t_n)}{2} \end{pmatrix}$$

$$\underline{A} \quad x^{(n+1)} = \underline{B} \quad x^{(n)} + \underline{C}$$

Starting with $n=0$ (IC), one computes

$\zeta^{(1)}$ and $\eta^{(1)}$ (solve linear syst. "$Ax=b$")

and loop,

$\leadsto$ $\zeta^{(n)} \approx \zeta(t_n)$ for $t_n = n \cdot k$ grid time

Finally, we get approx $u_h^n(x,t_n) = \sum_{j=1}^{m} \zeta_j^{(n)} \varphi_j(x)$

$\approx u(x,t_n)$ sol. wave eq.

Th: For the homogeneous wave eq. ($f=0$ in (W))
C-N preserves a discrete energy.
Forward / backward Euler NOT!!

Proof:

- C-N reads

multiply

$$M\left(\frac{\zeta^{(n+1)}-\zeta^{(n)}}{k}\right) - M\left(\frac{\eta^{(n+1)}+\eta^{(n)}}{2}\right) = 0 \quad \| \cdot k\left(\zeta^{(n+1)}+\zeta^{(n)}\right)^T S M^{-1}$$

$$M\left(\frac{\eta^{(n+1)}-\eta^{(n)}}{k}\right) + S\left(\frac{\zeta^{(n+1)}+\zeta^{(n)}}{2}\right) = 0 \quad \| \cdot k\left(\eta^{(n+1)}+\eta^{(n)}\right)^T$$

\+ add

$$\left(\zeta^{(n+1)}+\zeta^{(n)}\right)^T S\left(\zeta^{(n+1)}-\zeta^{(n)}\right) - \frac{k}{2}\cancel{\left(\zeta^{(n+1)}+\zeta^{(n)}\right)^T S\left(\eta^{(n+1)}+\eta^{(n)}\right)}$$

$$+\left(\eta^{(n+1)}+\eta^{(n)}\right)^T M\left(\eta^{(n+1)}-\eta^{(n)}\right) + \frac{k}{2}\cancel{\left(\eta^{(n+1)}+\eta^{(n)}\right)^T S\left(\zeta^{(n+1)}+\zeta^{(n)}\right)}$$

$$= 0$$

since S is symmetric