13) Show that the minimization problem

$$\left[\begin{array}{l} \text{Find } u_h \in V_h \text{ s.t. } F(u_h) \le F(v) \quad \forall\, v \in V_h \\ \text{where } \quad F(w) = \frac{1}{2}\int_0^1 a(w')^2\,dx - \int_0^1 f w\,dx \end{array}\right]$$

takes the matrix form

$$\left[\begin{array}{l} \text{Find } \chi = (\chi_j) \in \mathbb{R}^M \text{ that minimizes} \\ \frac{1}{2}\eta^T A \eta - b^T \eta \quad \text{for} \quad \eta \in \mathbb{R}^M \end{array}\right]$$

Solution

$V_h = \text{span}\{\varphi_j,\ j = 1, \dots, M\}$. Let $w = \sum_{j=1}^{M} \eta_j \varphi_j$.

We then have that

$$F(w) = \frac{1}{2}\int_0^1 a\left(\sum_{j=1}^{M} \eta_j \varphi_j'\right)^2 dx - \int_0^1 f \sum_{j=1}^{M} \eta_j \varphi_j\,dx =$$

$$= \left\{ \left(\sum_{j=1}^{M} \eta_j \varphi_j'\right)^2 = \sum_{j=1}^{M}\sum_{i=1}^{M} \eta_j \varphi_j' \cdot \eta_i \varphi_i' \right\} =$$

$$= \frac{1}{2}\sum_{j=1}^{M}\sum_{i=1}^{M} \eta_i \eta_j \int_0^1 a\,\varphi_j' \varphi_i'\,dx - \sum_{j=1}^{M} \eta_j \int_0^1 f\varphi_j\,dx =$$

$$= \left\{ \begin{array}{l} \text{For any vector } x \text{ and matrix } A: \\ [x_1 \dots x_M] \begin{bmatrix} a_{11} & \dots & a_{1M} \\ \vdots & & \\ a_{M1} & \dots & a_{MM} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix} = \dots = \sum_{i=1}^{M}\sum_{j=1}^{M} x_i a_{ij} x_j \end{array} \right\} =$$

$$= \frac{1}{2}\eta^T A \eta - b^T \eta \qquad b = \left(\int_0^1 f\varphi_j\,dx\right)_{j=1}^{M}$$

where $\eta = (\eta_i)_1^M$, $A = \left(\int_0^1 a\,\varphi_j' \varphi_i'\right)_{i,j=1}^{M}$ ⁄.

14) Consider the Neumann problem

$$\begin{cases} -(a(x)u'(x))' = f(x) & 0 < x \leq 1 \\ u(0)=0, \quad a(1)u'(1) = g_1 \end{cases}$$

The discrete variational formulation reads:

Find $U \in V_h$, $V_h = \{ v \in C(0,1), \text{ p.w. linear w.r.t } \mathcal{T}_h, v(0)=0 \}$

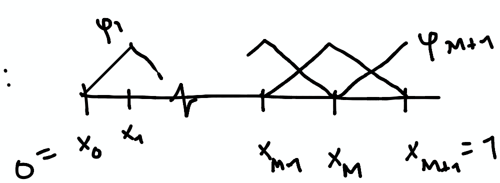
$\mathcal{T}_h$ partition $\{x_i\}_{i=0}^{M+1}$, such that

$$(*) \quad -g_1 v(1) + \int_0^1 a U' v' \, dx = \int_0^1 f v \, dx \quad \forall v \in V_h$$

* Compute stiffness matrix using uniform partition and assuming $f = a = g_1 = 1$.

* Check if the discrete equation corresponding to the test function φ_{M+1} at $x=1$ looks like a discrete analogue of the Neumann condition.

Solution V_h:



$U = \sum_{i=1}^{M+1} \xi_i \varphi_i$ and with $a = g_1 = f = 1$, $(*)$ turns into

$$\int_0^1 U'v' - v(1) = \int_0^1 v \quad \Rightarrow \quad \int_0^1 U'v' = \int_0^1 v + v(1) \quad \Rightarrow$$

$$\sum_{i=1}^{M+1} \xi_i \int_0^1 \varphi_i' \varphi_j' = \int_0^1 \varphi_j + \varphi_j(1)$$

For $i = 1, \dots, M$: $\varphi_i(x) = \frac{1}{h}\begin{cases} x - x_{i-1} & x \in [x_{i-1}, x_i] \\ x_{i+1} - x & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases}$ $\quad \varphi_i' = \begin{cases} 1/h \\ -1/h \\ 0 \end{cases}$

For $i = M+1$: $\varphi_{M+1}(x) = \frac{1}{h}\begin{cases} x - x_M & x \in [x_M, x_{M+1}] \\ 0 & \text{else} \end{cases}$ $\quad \varphi_{M+1}' = \begin{cases} 1/h \\ 0 \end{cases}$

Stiffness matrix: $(M+1)\times(M+1)$ $\quad \left(\int_0^1 \varphi_i' \varphi_j' \right)_{i,j=1}^{M+1}$

For $i=j$, $i = 1, \dots, M$: $\int_0^1 \varphi_i' \varphi_j' = \frac{2}{h}$

For $i=j=M+1$: $\int_0^1 \varphi_{M+1}' \varphi_{M+1}' = \frac{1}{h}$

For $j = i+1$: $\int_0^1 \varphi_i' \varphi_{i+1}' = -\frac{1}{h}$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & & \\ -1 & 2 & -1 & 0 & \cdots & \\ & \vdots & & & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \xi_M \\ \xi_{M+1} \end{bmatrix}$$

Discrete equation corr. to φ_{M+1}:

$$b_{M+1} = \int_0^1 \varphi_{M+1}\,dx + \varphi_{M+1}(1) = \int_{x_M}^{x_{M+1}} \frac{x - x_M}{h}\,dx + 1 = \frac{h}{2} + 1$$

$\Rightarrow$ Discrete equation:

$$-\frac{1}{h}\xi_M + \frac{1}{h}\xi_{M+1} = 1 + \frac{h}{2}$$

$$\frac{1}{h}(\xi_{M+1} - \xi_M) = 1 + \frac{h}{2}$$

$$\left\{ \begin{array}{l} U(x_i) = \xi_i \varphi_i = \xi_i \\ h = x_{i+1} - x_i \end{array} \right\}$$

$$\frac{U(x_{M+1}) - U(x_M)}{x_{M+1} - x_M} = 1 + \frac{x_{M+1} - x_M}{2}$$

Let $M \to \infty$ (finer and finer grid) $\Rightarrow U'(1) = 1$.

$\because$ Yes, the discrete equation looks like the Neumann boundary condition. $\%$

18) Prove an a priori and an a posteriori error estimate for the cG(1) method applied to the BVP

$$\begin{cases} -u'' + bu' + u = f & \text{in } (0,1) \quad b \text{ constant} \\ u(0) = u(1) = 0 \end{cases}$$

<u>Solution</u>

$$\left[\begin{array}{ll} \text{Recall: A priori:} & \|u-u_h\|_E \leq \text{smth}(\|u''\|_{L^2}) \\ \text{A posteriori:} & \|u-u_h\|_E \leq \text{smth}(\|R(u_h)\|_{L^2}) \\ & R(u_h) = f + u_h'' - bu_h' - u_h \end{array}\right]$$

Variational formulation:

$$\int_0^1 u'v' + \int_0^1 bu'v + \int_0^1 uv = \int_0^1 fv \qquad \forall v \in H_0^1(0,1)$$

cG(1): Find $u_h \in V_h = \{v \in H_0^1(0,1) : \text{p.w. linear on grid}\}$ s.t.

$$\int_0^1 u_h'v' + \int_0^1 bu_h'v + \int_0^1 u_h v = \int_0^1 fv \qquad \forall v \in V_h$$

<u>G⊥</u>: $\displaystyle\int_0^1 \left((u-u_h)'v' + b(u-u_h)'v + (u-u_h)v\right) dx = 0 \qquad \forall v \in V_h$

$$a(u-u_h, v) = 0 \quad \forall v \in V_h$$

Note that $\int_0^1 ((v')^2 + \underbrace{bv'v}_{=\frac{b}{2}\frac{d}{dx}(v^2)} + v^2) = \int_0^1 ((v')^2 + v^2) + \frac{b}{2}[v^2]_0^1 =$

$$= \{v \in H_0^1\} = \int_0^1 ((v')^2 + v^2) = \|v\|^2_{H^1(0,1)}$$

<u>A priori</u> Let $e = u - u_h$.

$$\|e\|^2_{H^1(0,1)} = \int_0^1 ((e')^2 + be'e + e^2)\,dx =$$

$$= \int_0^1 (e'(u-u_h)' + be'(u-u_h) + e(u-u_h))\,dx = \{v \in V_h\}.$$

$$= \int_0^1 (e'(u-v)' + be'(u-v) + e(u-v))dx + \underbrace{\int_0^1 (e'(v-u_h)' + be'(v-u_h) + e(v-u_h))dx}_{=0 \quad (G\perp,\ v-u_h \in V_h)}$$

$$= \int_0^1 (e'(u-v)' + be'(u-v) + e(u-v))\,dx =$$

$$= (e', (u-v)') + (e, u-v) + (be', u-v) \leq \{C\text{-}S\} \leq$$

$$\leq |e|_{H^1}|u-v|_{H^1} + \|e\|_{L^2}\|u-v\|_{L^2} + b\,|e|_{H^1}\|u-v\|_{L^2} \leq \left\{a+b \leq \sqrt{2}(a^2+b^2)^{1/2}\right\}$$

$$\leq \sqrt{2}\,\|e\|_{H^1}\|u-v\|_{H^1} + b\|e\|_{H^1}\|u-v\|_{H^1} = (\sqrt{2}+b)\|e\|_{H^1}\|u-v\|_{H^1}$$

$$\Rightarrow \|e\|_{H^1(0,1)} \leq (\sqrt{2}+b)\|u-v\|_{H^1} \qquad \left\{ \begin{array}{l} \|\cdot\|^2_{H^1} = \|\cdot\|^2_{L^2} + |\cdot|^2_{H^1} \\ |v|^2_{H^1} = \|\nabla v\|^2_{L^2} \end{array} \right\}$$

Let $v = \pi_1 u$

$$\|u-v\|^2_{H^1} = \|u-v\|^2_{L^2} + \|(u-v)'\|^2_{L^2}$$

$$\|u-\pi_1 u\|^2_{H^1} = \|u-\pi_1 u\|^2_{L^2} + \|(u-\pi_1 u)'\|^2_{L^2} \overset{\text{thm 3.2}}{\leq} (Ch^2+Dh)^2\|u''\|^2_{L^2}$$

$$\Rightarrow \|e\|_{H^1} \leq (\sqrt{2}+b)(Ch^2+Dh)\|u''\|_{L^2}.$$

∴

A posteriori

$$\|e\|_{H^1}^2 = \int_0^1 ((e')^2 + be'e + e^2)\,dx =$$

$$= \underbrace{\int_0^1 (u'e' + bu'e + ue)\,dx}_{= \int_0^1 fe\,dx \text{ since } u \text{ is the exact solution}} - \int_0^1 (u_h'e' + bu_h'e + u_h e)\,dx = \left\{ \begin{array}{l} \pi_1 e \in V_h \\ u_h = \text{sol. in } V_h \end{array} \right\}$$

$$= \int_0^1 fe\,dx - \int_0^1 (u_h'e' + bu_h'e + u_h e)\,dx + \underbrace{\int_0^1 (u_h'(\pi_1 e)' + bu_h'(\pi_1 e) + u_h \pi_1 e)\,dx - \int_0^1 f\pi_1 e\,dx}_{=0}$$

$$= \int_0^1 f(e - \pi_1 e)\,dx - \int_0^1 (\underbrace{u_h'(e-\pi_1 e)'}_{\text{want } u_h'' \text{ back}} + bu_h'(e - \pi_1 e) + u_h(e - \pi_1 e))\,dx$$

want u_h'' back
Split into subintervals and integrate by parts.

$$= \int_0^1 f(e - \pi_1 e)\,dx - \int_0^1 (bu_h' + u_h)(e - \pi_1 e)\,dx + \sum \int_{x_{j-1}}^{x_j} u_h''(e - \pi_1 e)\,dx =$$

$$= \int_0^1 \underbrace{(f + u_h'' - bu_h' - u_h)}_{R(u_h)}(e - \pi_1 e)\,dx = \int_0^1 h\,R(u_h)\,\frac{e - \pi_1 e}{h}\,dx \leq$$

$$\leq \{C\text{-}S\} \leq \|hR(u_h)\|_{L^2(0,1)} \left\| \frac{e - \pi_1 e}{h} \right\|_{L^2(0,1)} \leq \left\{ \begin{array}{l} \text{thm } 3.2 \\ \left\|\frac{e - \pi_1 e}{h}\right\|_{L^2(I_i)} \leq C\|e'\|_{L^2(I_i)} \end{array} \right\}$$

$$\leq C\|hR(u_h)\|_{L^2(0,1)} \|e'\|_{L^2(0,1)} \leq C\|hR(u_h)\|_{L^2(0,1)} \|e\|_{H^1(0,1)}$$

$$\Rightarrow \|e\|_{H^1} \leq Ch\|R(u_h)\|_{L^2(0,1)}.$$

15) Consider the Robin problem

$$\begin{cases} -(a(x)\,u'(x))' = f(x) & 0<x<1 \\ u(0)=0, \quad a(1)u'(1) + \gamma(u(1)-u_1) = g_1 \end{cases}$$

$\gamma>0$ bound. heat conduct.

u_1 given outside temp.

g_1 given heat flux

Variational formulation?

Solution.

Multiply DE by $v \in V = \{v \in H^1(0,1) : v(0)=0\}$, integrate

$$-\int_0^1 (a(x)\,u'(x))'\,v(x) = \int_0^1 f v\,dx$$

$$-[a(x)u'(x)\,v(x)]_0^1 + \int_0^1 a(x)u'(x)v'(x) = \int_0^1 fv\,dx$$

$$\underbrace{-a(1)u'(1)\,v(1)}_{=g_1-\gamma(u(1)-u_1)} + \underbrace{a(0)\,u'(0)\,v(0)}_{\substack{=0 \\ \text{since } v(0)=0}} + \int_0^1 au'v'\,dx = \int_0^1 fv\,dx$$

$$-g_1 v(1) + \gamma u(1)v(1) - \gamma u_1 v(1) + \int_0^1 au'v'\,dx = \int_0^1 fv\,dx$$

$\Rightarrow$ VF: Find $u \in V$ s.t.

$$\int_0^1 au'v'\,dx + \gamma u(1)v(1) = \int_0^1 fv\,dx + g_1 v(1) + \gamma u_1 v(1).$$

17) Prove the following inequality for $v(x)$ on $I=(0,1)$
with $v(0)=0$.

$$v(y) = \int_0^y v'(x)\,dx \le \left(\int_0^y \frac{1}{a}\,dx\right)^{1/2}\left(\int_0^y a(v')^2\,dx\right)^{1/2}, \quad y \in I,$$

$a(x) > 0$

Use this inequality to show that if $\frac{1}{a} \in L^1(I)$,
then $\|v\|_E$ being small, and $v(0)=0 \Rightarrow \|v\|_\infty$ is small.

Solution

$$\int_0^y v'(x)\,dx = \int_0^y \left(\frac{1}{a}\right)^{1/2} a^{1/2} \cdot v'\,dx \le \{C\text{-}S\} \le$$

$$\le \left(\int_0^y \frac{1}{a}\,dx\right)^{1/2}\left(\int_0^y a(v')^2\,dx\right)^{1/2}$$

Assume $a^{-1} \in L^1(I)$. Show $\|v\|_E < \varepsilon$, $v(0)=0 \Rightarrow \|v\|_\infty < C\varepsilon$
or similar.

$$\|v\|_E^2 = \int_0^1 a(x)\,|v'(x)|^2\,dx$$

$$|v(y)| = \left|\int_0^y v'(x)\,dx\right| \le \left(\int_0^y \frac{1}{a}\,dx\right)^{1/2}\left(\int_0^y a(v')^2\,dx\right)^{2} \le \{a^{-1} \in L^1\}$$

$$\le C\left(\int_0^y a(v')^2\,dx\right)^{1/2} \le C\|v\|_E \le C\varepsilon.$$

7.5) Show that for $v \in C^1(0,1)$:

$$\|v\|^2 \leq v(0)^2 + v(1)^2 + \|v'\|^2$$

Solution

$$\|v\|^2 = \int_0^1 v^2 dx = \int_0^{1/2} v^2(x) dx + \int_{1/2}^1 v^2(x) dx = \left\{ \begin{array}{l} \text{P.I.} \\ \frac{d}{dx}(x-\frac{1}{2}) = 1 \end{array} \right\} =$$

$$= \left[(x-\tfrac{1}{2}) v^2(x)\right]_0^{1/2} + \left[(x-\tfrac{1}{2}) v^2(x)\right]_{1/2}^1 - \int_0^1 (x-\tfrac{1}{2}) 2v'(x) v(x) dx \leq$$

$$\leq \{2ab \leq a^2 + b^2\} \leq \frac{1}{2}v^2(0) + \frac{1}{2}v^2(1) + \int_0^1 \frac{1}{2}(v')^2 + \int_0^1 2\underbrace{(x-\tfrac{1}{2})^2}_{\leq \frac{1}{4}} v^2 dx$$

$$\Rightarrow \frac{1}{2}\|v\|^2 \leq \frac{1}{2}v^2(0) + \frac{1}{2}v^2(1) + \frac{1}{2}\|v'\|^2_{L^2(0,1)}$$

//.