Chapter 10: FEM for Poisson's equation in $2d$ (summary)

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Goal: Derive FEM for Poisson's equation and give error estimates.

- Let $\Omega \subset \mathbb{R}^2$ be a domain with polygonal boundary and $f : \Omega \to \mathbb{R}$ a nice function. Consider Poisson's equation

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The notation $\nabla^2 u$ is sometimes used for the Laplacian.

The variational formulation of the above PDE reads

$$\text{Find} \quad u \in H_0^1(\Omega) \quad \text{such that} \quad (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

- Given a triangulation T_h of Ω, one defines the space

$$V_h^0(\Omega) = \{v \in V_h : v_{|\partial\Omega} = 0\} = \mathrm{span}(\{\varphi_j\}_{j=1}^{n_i}),$$

where φ_j are hat functions and n_i denotes the number of interior nodes.

The finite element problem for Poisson's equation then reads

$$\text{Find} \quad u_h \in V_h^0(\Omega) \quad \text{such that} \quad (\nabla u_h, \nabla v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h^0(\Omega).$$

As in the case of BVP, the FE problem then yields the linear system of equations

$$S\zeta = b,$$

where the stiffness matrix S has entries $s_{i,j} = (\nabla\varphi_j, \nabla\varphi_i)_{L^2(\Omega)}$, for $i, j = 1, \ldots, n_i$, the load vector has components $b_i = (f, \varphi_i)_{L^2(\Omega)}$ for $i = 1, \ldots, n_i$, and the unknown vector ζ provides the finite element solution $u_h = \sum_{j=1}^{n_i} \zeta_j \varphi_j$ which is a numerical approximation of the exact solution u to Poisson's equation.

- A FE code for Poisson's equation in $2d$ needs the following:

A point matrix listening all nodes of the mesh of the domain Ω, a connectivity matrix containing all triangles of the mesh as well as information related to the real boundary of the domain $\partial\Omega$.

An assembly procedure in order to compute the stiffness matrix S using all element stiffness matrices.

A procedure to compute the element stiffness matrix using a linear map and the reference triangle.

A procedure to compute the element load vector using a linear map and the reference triangle.

- We state Poincaré's inequality in $2d$: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then, there exists a constant $C > 0$ such that

$$\|v\|_{L^2(\Omega)} \le C \|\nabla v\|_{L^2(\Omega)}$$

for all $v \in H_0^1(\Omega)$.

- Galerkin orthogonality condition in the present context reads: Let u and u_h denote the solutions to the VF and FE problems and assume that they are smooth enough. Then, one has

  $$\int_\Omega \nabla(u-u_h)\cdot\nabla v_n \, \mathrm{d}x = 0$$

  for all $v_h \in V_h^0(\Omega)$.

  Galerkin's orthogonality is used to show that the FE solution u_h is the best approximation of u in $V_h^0(\Omega)$ in the energy norm:

  $$\|u-u_h\|_E \leq \|u-v\|_E$$

  for all $v \in V_h^0(\Omega)$. Here, we recall that the energy norm reads $\|v\|_E = \|\nabla v\|_{L^2(\Omega)}$.

  The above is then used to show an a priori error estimate in the energy norm for Poisson's equation: Let u and u_h denote the solutions to the VF and FE problems. Under some technical assumptions, one has the following error estimate

  $$\|u-u_h\|_E \leq Ch\|u\|_{H^2(\Omega)}\,.$$

  This directly implies

  $$\|u-u_h\|_{L^2(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}\,.$$

  Note that, a further analysis of FEM provides the optimal a priori error estimate in the L^2-norm

  $$\|u-u_h\|_{L^2(\Omega)} \leq Ch^2\|u\|_{H^2(\Omega)}\,.$$

Further resources:

- VF for Poisson eq. at wikiversity.org
- FEM for Poisson eq. at wikiversity.org
- FEM for Poisson eq. at math.uci.edu
- FEM for Poisson eq. at fenicsproject.org
- FEM at github.io
- Assembly at caendkoelsch.wordpress.com