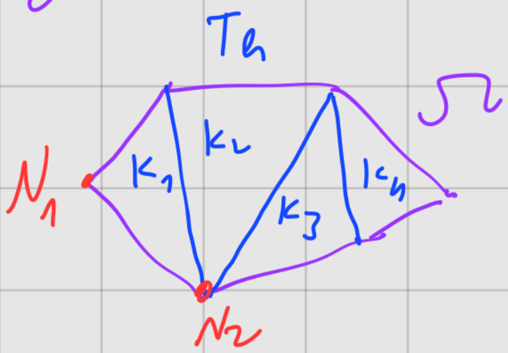
Ⓡ • Poisson $\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \quad (d=2) \\ u = 0 & \text{on } \partial\Omega \end{cases}$

• Green $\to$ VF

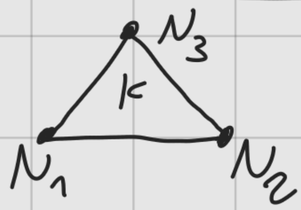
• Triangulation $T_h = \{K\}$ K triangle



$N_j \to$ nodes

4) The space of linear polynomials:

Let K be a triangle



Def: The space of linear fct on K is defined by

$$P_1(K) = \{ v : K \to \mathbb{R} : v(x_1, x_2) = c_0 + c_1 x_1 + c_2 x_2 \text{ , for } x_1, x_2 \in \mathbb{R}, \ c_0, c_1, c_2 \in \mathbb{R} \}$$

Rem: Every $v \in P_1(K)$ is uniquely determined by its values at the nodes. That is

$\alpha_j := v(N_j) = v(x_1^{(j)}, x_2^{(j)})$, for $j = 1,2,3$.

Indeed:

$$\alpha_1 = v(N_1) = c_0 + c_1 x_1^{(1)} + c_2 x_2^{(1)}$$

$$\alpha_2 = v(N_2) = c_0 + c_1 x_1^{(2)} + c_2 x_2^{(2)}$$

$$\alpha_3 = v(N_3) = c_0 + c_1 x_1^{(3)} + c_2 x_2^{(3)}$$

$\hookrightarrow$ This is a linear syst of 3 eq. for 3 unknows c_0, c_1, c_2

$\to \exists!$ sol. (K non degenerate)

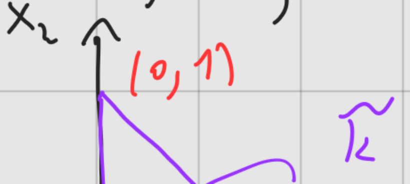
One preferes to work with a nodal basis, $\{\lambda_1, \lambda_2, \lambda_3\}$, "defined" by

$$\lambda_j(N_i) = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases} \qquad (\text{for } i,j = 1,2,3)$$

That each $v \in P_1(K)$ can be written as

$$v = \sum_{j=1}^{3} \alpha_j \lambda_j \quad , \text{ where } \alpha_j = v(N_j) \text{ for } j = 1,2,3.$$

For the reference triangle $\tilde{K}$ with nodes $(0,0)$, $(1,0)$, $(0,1)$:



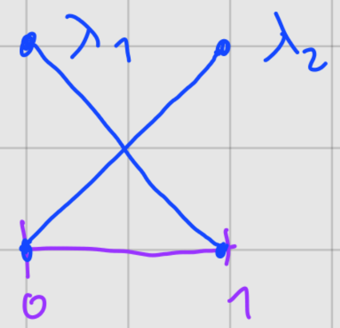
One has

$\{\lambda_1(x_1, x_2) = 1 - x_1 - x_2$

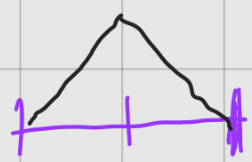
$$\begin{cases} \lambda_2(x_1,x_2) = x_1 \\ \lambda_3(x_1,x_2) = x_2 \end{cases}$$

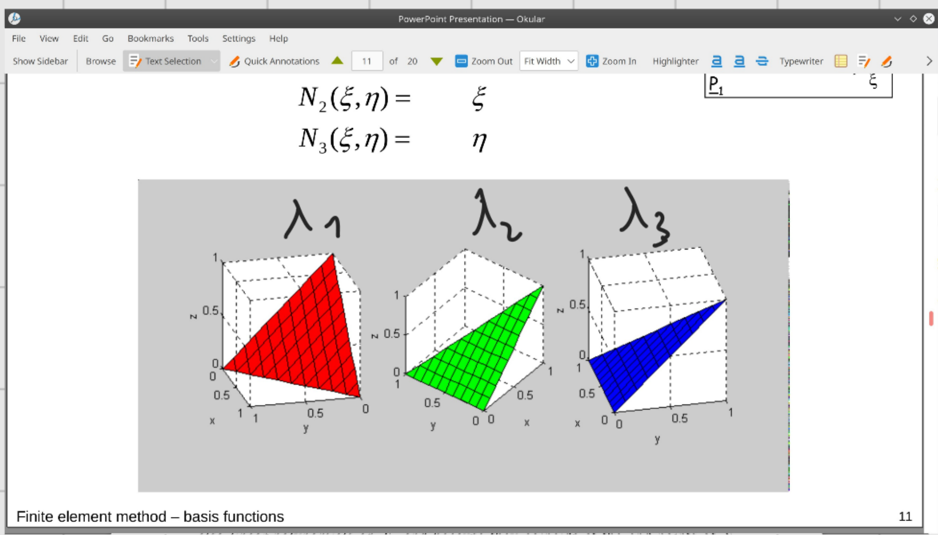
Indeed, $\lambda_1((0,0)) = 1$, $\lambda_1((1,0)) = 0$, $\lambda_1((0,1)) = 0 \to$ OK and "same" is true for λ_2, λ_3.

Rem:



in 1d





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5) The space of continuous pw linear polynomials:

Let $T_h = \{K\}$ be a mesh / triangulation of of a nice domain $\Omega \subset \mathbb{R}^2$ with polygonal $\partial\Omega$.

Def: The space of continuous piecewise linear polynomials is defined by

$$V_h = \{ v \in C^0(\Omega) : v|_K \in P_1(K) \ \forall K \in T_h \}$$

continuous

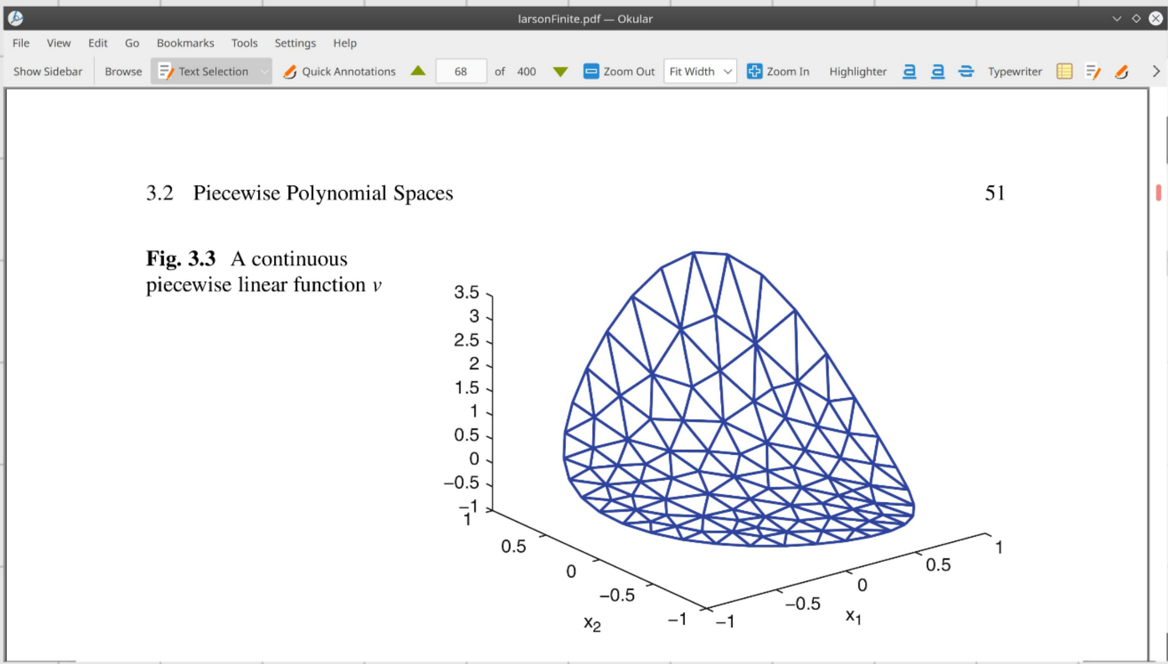

Fig. 3.3 A continuous piecewise linear function v

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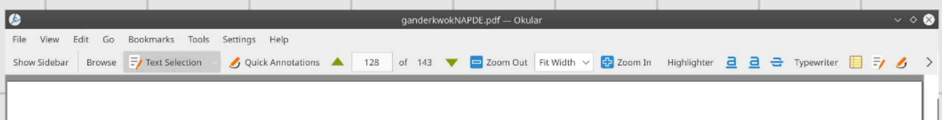
As in 1d, the above space has the hat functions $\{\varphi_j\}_{j=1}^{n_p}$ as a basis.

Here n_p = number of nodes in $T_h = \{K\}$.
Once again, one has

$$\varphi_j(N_i) = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases} \quad \text{for } i, j = 1, \dots, n_p.$$



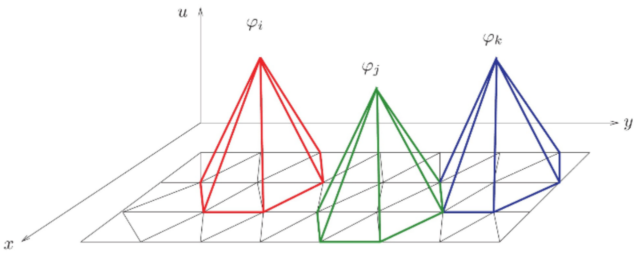


Figure 5.4. *Examples of hat functions in two dimensions.*

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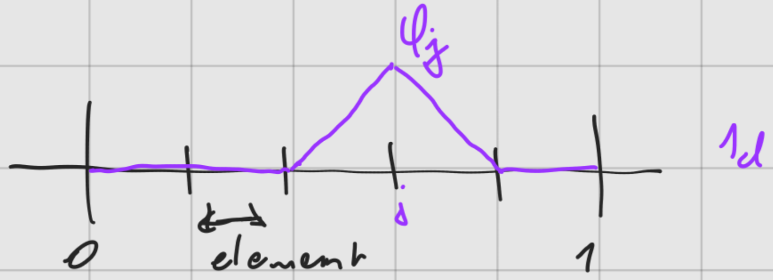


⚠ As in 1d, each fct in V_h can written in terms of φ_j:

$$v = \sum_{j=1}^{n_p} \alpha_j \varphi_j \quad , \text{ where } \alpha_j = v(N_j) \text{ for } j = 1, 2, 3, \dots, n_p$$

Rem: The same works in 3d, f.ex. using thetrahedra.

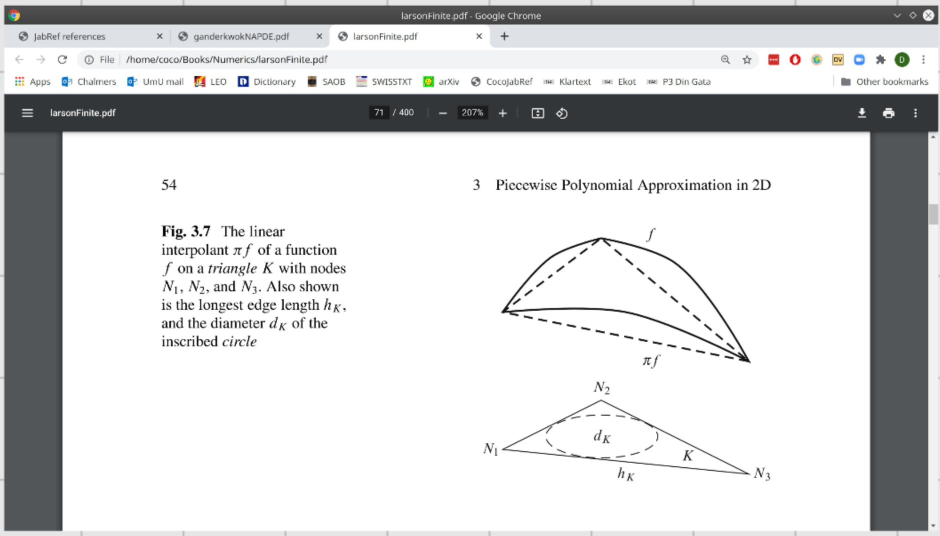
6) Linear interpolation:

Consider a continuous fct f on a triangle K with nodes N_1, N_2, N_3.

Def: The linear interpolant of f is $\pi_1 f \in P_1(K)$ is defined by:

$$\pi_1 f = \sum_{j=1}^{3} f(N_j)\varphi_j$$

Rem: $\pi_1 f$ is a plane:



(c) Larson – Bengzon

For error of such interpolant, one has the following result:

Th: For $f \in H^2(K)$, one has

$$\| \pi_1 f - f \|_{L^2(K)} \leq C_K' \, h_K^{2} \, \| f \|_{H^2(K)}$$

$$\| \nabla (f - \pi_1 f) \|_{L^2(K)} \leq C_K \cdot h_K^{1} \cdot \| f \|_{H^2(K)},$$

constant that depends on triangle K.

where h_K : is the length of the longest edge of K

7) Continuous pw linear interpolant:

Let $\Omega \subset \mathbb{R}^2$ polygonal domain and $f : \Omega \to \mathbb{R}$ continuous.

Let $T_h = \{K\}$ be a (nice) triangulation of Ω and

V_h the space of cont. pw. linear fct on T_h.

Def: The continuous piecewise linear interpolant of f is $\Pi_h f \in V_h$ is defined by

$$\Pi_h f = \sum_{j=1}^{n_p} f(N_j)\varphi_j \,,$$

where φ_j are hat fct, n_p = nbr of nodes in T_h.

Rem: pdesurf in Matlab.

The error of such interpolant is given by

Th: Let $\Omega \subset \mathbb{R}^2$ polygonal domain. Let $f \in H^2(\Omega)$ and its interpolant $\Pi_h f \in V_h$. Then, one has the estimates

$$\| f - \Pi_h f \|^2_{L^2(\Omega)} \leq C \cdot \sum_{K \in T_h} h_K^4 \cdot \| f \|^2_{H^2(K)}$$

$$\| \nabla (f - \Pi_h f) \|^2_{L^2(\Omega)} \leq C \cdot \sum_{K \in T_h} h_K^2 \, \| f \|^2_{H^2(K)}$$

Proof:

We have

previous theorem

$$\| f - \Pi_h f \|^2_{L^2(\Omega)} = \sum_{K \in T_h} \| f - \Pi_h f \|^2_{L^2(K)} \leq \sum_{K \in T_h} C_K^2 \cdot h_K^4 \cdot \| f \|^2_{H^2(K)} \leq$$

$\Omega = \bigcup_{K \in T_h} K$

$\Pi_h f = \Pi_1 f$ since $\Pi_h f$ is linear on K.

$$\leq C^2 \cdot \sum_{K \in T_h} h_K^4 \cdot \| f \|^2_{H^2(K)} \leq$$

for nice T_h → one

can bound all $c_K \le C$

$$\le C^2 \cdot h^4 \cdot \sum_{K \in T_h} \|f\|^2_{H^2(K)} \le C^2 \cdot \boxed{h^4} \cdot \|f\|^2_{H^2(\Omega)}$$

$h := \max_{K \in T_h} |h_K|$

Def of T_h

$\blacksquare$

<u>Rem</u>: Let $r \geq 2$ (integer). Errors for pw polynomials of degree $r-1$ is more difficult to show. One has

$$\|\Pi_h f - f\|_{L^2(\Omega)} \le C \cdot h^r \cdot \|f\|_{H^r(\Omega)} \text{ for } f \in H^r(\Omega).$$