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Chapter 12: FEM for wave equations in higher dimensions (summary)

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Goal: Study the exact solution to the wave equation (conservation of energy), derive a FE discretisation for this PDE and provide a semi-discrete error estimate for the FEM.

- Let $\Omega \subset \mathbb{R}^d$ be a nice domain with smooth boundary. Consider the homogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+ \\ u(\cdot,0) = u_0 \quad \text{in} \quad \Omega \\ u_t(\cdot,0) = v_0 \quad \text{in} \quad \Omega. \end{cases}$$

  One has conservation of energy for solutions to the above PDE:

$$\frac{1}{2}\left(\|u_t(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla u(\cdot,t)\|^2_{L^2(\Omega)}\right) = \frac{1}{2}\left(\|v_0\|^2_{L^2(\Omega)} + \|\nabla u_0\|^2_{L^2(\Omega)}\right) = \text{Const} \quad \forall t > 0.$$

- The variational formulation of the inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f \quad \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+ \\ u(\cdot,0) = u_0 \quad \text{in} \quad \Omega \\ u_t(\cdot,0) = v_0 \quad \text{in} \quad \Omega. \end{cases}$$

  reads: For each $t > 0$ of interest,

$$\text{Find} \quad u(\cdot,t) \in H^1_0(\Omega) \quad \text{such that} \quad (u_{tt}, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega)$$

  and $u(\cdot,0) = u_0, u_t(\cdot,0) = v_0$ in Ω for the initial values.

- Let now T_h denotes a mesh of Ω and V_h the space of continuous piecewise linear functions of T_h. Consider the space $V_h^0 = V_h^0(\Omega) = \{v\colon \Omega \to \mathbb{R} : v \text{ continuous pw linear on } T_h \text{ and } v = 0 \text{ on } \partial\Omega\}$ and observe that $V_h^0 = \text{span}(\{\varphi_j\}_{j=1}^{n_i})$, where n_i denotes the number of interior nodes. The finite element problem for the above wave equation reads: For each $t > 0$ of interest,

$$\text{Find} \quad u_h(\cdot,t) \in V_h^0(\Omega) \quad \text{such that} \quad (u_{h,tt}, v_h)_{L^2(\Omega)} + (\nabla u_h, \nabla v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h^0(\Omega)$$

  and $u_h(x,0) = \pi_h u_0(x), u_{h,t}(x,0) = \pi_h v_0(x)$ in Ω for the initial values.

  As always, writing $u_h(x,t) = \sum_{j=1}^{n_i} \zeta_j(t)\varphi_j(x)$ and taking $v_h = \varphi_i$ in the FE problem gives the system of linear ODEs

$$\begin{cases} M\ddot{\zeta}(t) + S\zeta(t) = F(t) \\ \zeta(0) = \zeta_0, \dot{\zeta}(0) = \eta_0. \end{cases}$$

  One has the following a priori error estimate for the FE approximation of the inhomogeneous wave equation

$$\|u_h(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \leq C\left(|\pi_h u_0 - R_h u_0|_1 + \|\pi_h v_0 - v_0\|_{L^2(\Omega)}\right) + Ch^2\left(\|u(t)\|_{H^2(\Omega)} + \int_0^t \|u_{tt}(\cdot,s)\|_{H^2(\Omega)}\,\mathrm{d}s\right),$$

  where we recall that R_h denotes Ritz projection.