21) Consider the IVP

$$\begin{cases} \dot{u}(t) + a(t)u(t) = f(t), & 0<t<T \\ u(0) = u_0 \end{cases}$$

Assume that $\int_{I_j} f(s)\,ds = 0$, $j = 1, 2, \dots$, where $I_j = (t_{j-1}, t_j)$, $t_j = kj$, $k>0$. Prove that if $a(t) \geq 0$ then solution u satisfies

$$|u(t)| \leq e^{-A(t)}|u_0| + \max_{0\leq s\leq t} |kf(s)|.$$

Solution (cf. thm 6.2)

From thm 6.1 we have the solution formula:

$$u(t) = u_0 e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))} f(s)\,ds$$

$$a(t) = A'(t) \geq 0 \Rightarrow A(t) \geq A(s), \text{ for } t \geq s$$

$$\Rightarrow e^{-(A(t)-A(s))} \leq 1$$

$$\Rightarrow |u(t)| \leq |u_0| e^{-A(t)} + \left|\int_0^t f(s)\,ds\right|$$

Let $t \in I_j$. Then we have

$$\left|\int_0^t f(s)\,ds\right| = \left|\sum_{i=1}^{j-1} \underbrace{\int_{I_i} f(s)\,ds}_{=0} + \int_{t_{j-1}}^t f(s)\,ds\right| =$$

$$= \left|\int_{t_{j-1}}^t f(s)\,ds\right| \leq k\|f\|_{L^\infty(0,t)}$$

$$\Rightarrow |u(t)| \leq e^{-A(t)}|u_0| + \max_{0\leq s\leq t} |kf(s)|$$

22) Compute the cG(1) approximation for the IVP

$$\begin{cases} \dot{u}(t) + a(t)u(t) = f(t) & 0 < t \leq T \\ u(0) = u_0 \end{cases}$$

a) $a(t) = 4$, $f(t) = t^2$, $u_0 = 1$

b) $a(t) = -t$, $f(t) = t^2$, $u_0 = 1$

Solution

Divide $(0,T)$ into equidistant subintervals $I_n := (t_{n-1}, t_n)$, $t_n - t_{n-1} =: k$.

cG(1) method for IVP:s:

Trial space = p.w. linear functions

Basis: usual hat functions

Test space = p.w. constant functions

Basis: $v \equiv 1$ on each subinterval, i.e. $v_n = 1_{I_n}$

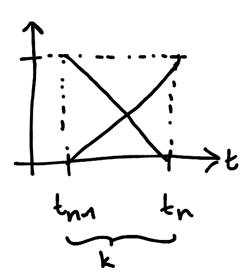
a) Multiply DE by test function and integrate:

$$\int_{t_{n-1}}^{t_n} \dot{u}(t)\,dt + 4\int_{t_{n-1}}^{t_n} u(t)\,dt = \int_{t_{n-1}}^{t_n} t^2\,dt \qquad (*)$$

On an interval (t_{n-1}, t_n), $u(t)$ is approximated by a p.w. linear fcn:

$$\int_{t_{n-1}}^{t_n} u(t)\,dt \approx \int_{t_{n-1}}^{t_n} \left(u_{n-1}\varphi_{n-1}(t) + u_n\varphi_n(t)\right) dt = \ldots = \frac{k}{2}(u_{n-1} + u_n)$$

φ_i usual hat fcns:



$$\int_{t_{n-1}}^{t_n} t^2\,dt = \ldots = \frac{1}{3}(t_n^3 - t_{n-1}^3) = \{t_n = nk\} = \ldots = \frac{3n^2 - 3n - 1}{3}k^3$$

$$(*) \Rightarrow \quad u_n - u_{n-1} + 2k(u_{n-1} + u_n) = \frac{3n^2 - 3n - 1}{3} k^3$$

$$(1+2k)u_n = (1-2k)u_{n-1} + \frac{3n^2-3n-1}{3} k^3$$

$$u_n = \frac{1-2k}{1+2k}\left(u_{n-1} + \frac{3n^2-3n-1}{3(1-2k)} k^3 \right) =$$

$$= \frac{1-2k}{1+2k}\left(\frac{1-2k}{1+2k}\left(u_{n-2} + \frac{3n^2-3n-1}{3(1-2k)} k^3 \right) + \frac{3n^2-3n-1}{3(1-2k)} k^3 \right) = \ldots =$$

$$= \left(\frac{1-2k}{1+2k}\right)^n u_0 + \sum_{j=0}^{n-1} \left(\frac{1-2k}{1+2k}\right)^{n-j} \frac{3n^2-3n-1}{3(1-2k)} k^3 = \{u_0 = 1\} =$$

$$= \underbrace{\left(\frac{1-2k}{1+2k}\right)^n}_{<1} + \sum_{j=0}^{n-1} \left(\frac{1-2k}{1+2k}\right)^{n-j} \frac{3n^2-3n-1}{3(1-2k)} k^3$$

Effect of initial data u_0 decays with time
cf. prob. 21 and/or thm 6.2a) $(a(t) = 4 > 0)$

b) Multiply DE by test function and integrate $\Rightarrow$

$$\int_{t_{n-1}}^{t_n} \dot{u}(t)\, dt - \underbrace{\int_{t_{n-1}}^{t_n} t\, u(t)\, dt}_{(***)} = \int_{t_{n-1}}^{t_n} t^2\, dt \qquad (**)$$

$$(***) = \int_{t_{n-1}}^{t_n} t\left(u_{n-1}\varphi_{n-1}(t) + u_n \varphi_n(t)\right) dt = \left\{ \begin{array}{ll} \varphi_{n-1}(t) = \frac{t_n - t}{k}, & t \in [t_{n-1}, t_n] \\ \varphi_n(t) = \frac{t - t_{n-1}}{k}, & t \in [t_{n-1}, t_n] \end{array} \right\}$$

$$= \frac{1}{k}\int_{t_{n-1}}^{t_n} \left(u_{n-1}(t t_n - t^2) + u_n(t^2 - t t_{n-1})\right) dt = \ldots =$$

$$= -\frac{k}{6}\left(2u_{n-1}k - 3u_{n-1}t_n + u_nk - 3u_nt_n\right) = \{t_n = nk\} =$$

$$= u_{n-1}\left(\frac{1}{2}nk^2 - \frac{1}{3}k^2\right) + u_n\left(\frac{1}{2}nk^2 - \frac{1}{6}k^2\right)$$

$$(**) \Rightarrow \quad u_n - u_{n-1} - u_{n-1}\left(\frac{1}{2}nk^2 - \frac{1}{3}k^2\right) - u_n\left(\frac{1}{2}nk^2 - \frac{1}{6}k^2\right) = \frac{3n^2-3n-1}{3}k^3$$

$$u_n\left(1 + \frac{1}{6}k^2 - \frac{1}{2}nk^2\right) - u_{n-1}\left(1 + \frac{1}{2}nk^2 - \frac{1}{3}k^2\right) = \frac{3n^2-3n-1}{3}k^3$$

$$\Rightarrow u_n = \frac{1+k^2\left(\frac{n}{2}-\frac{1}{3}\right)}{1-k^2\left(\frac{n}{2}-\frac{1}{6}\right)}\left(u_{n-1} + \frac{3n^2-3n-1}{3\left(1+k^2\left(\frac{n}{2}-\frac{1}{3}\right)\right)}k^3\right) = \ldots =$$

$$= \underbrace{\left(\frac{1+k^2\left(\frac{n}{2}-\frac{1}{3}\right)}{1-k^2\left(\frac{n}{2}-\frac{1}{6}\right)}\right)^n}_{>1} + \sum_{j=0}^{n-1}\left(\frac{1+k^2\left(\frac{n}{2}-\frac{1}{3}\right)}{1-k^2\left(\frac{n}{2}-\frac{1}{6}\right)}\right)^{n-j}\frac{3n^2-3n-1}{3\left(1+k^2\left(\frac{n}{2}-\frac{1}{3}\right)\right)}k^3$$

$\Rightarrow$ Effect of initial data increases with time $(a(t) = -t \leq 0)$

23) Consider the dG(0) discontinuous Galerkin method for the equation:

$$\begin{cases} \dot{u}(t) + au(t) = 0 & 0 < t \leq T, \quad a \geq 0 \text{ const.} \\ u(0) = u_0 \end{cases}$$

Prove the stability estimate:

$$|U_N|^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \leq |u_0|^2$$

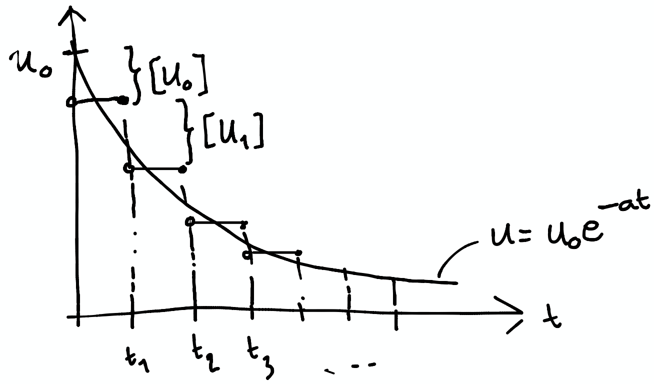
Solution

First some definitions:

$$U_n^+ := \lim_{t \downarrow t_n} U(t) = U\big|_{(t_n, t_{n+1}]} = U_{n+1}$$

$$[U_n] = U_n^+ - U_n^- = U_{n+1} - U_n$$

$$U_0 = u_0$$



$0 = t_0 < \dots < t_N = T$

$t_j - t_{j-1} = k$

dG(0) formulation: Find $U \in W_k^{(0)} = \{v : v \text{ p.w. const}\}$ s.t.

$$(*) \quad \sum_{n=1}^{N} \left(\int_{t_{n-1}}^{t_n} (\dot{U} + aU) v \, dt + [U_{n-1}] v_{n-1}^+ \right) = 0 \quad \forall v \in W_k^{(0)}.$$

For U p.w. const, we have that $\dot{U}\big|_{(t_{n-1}, t_n)} = 0, \quad n = 1, \dots, N$

Have that $U = \sum U_n 1_{I_n}$, where 1_{I_n} is the indicator fcn.

Basis fcns $v_n = 1_{I_n}$.

$$(*) \Rightarrow \int_{t_{n-1}}^{t_n} aU_n \, dt + U_n - U_{n-1} = 0 \qquad n = 1, \dots, N$$

$$\Rightarrow akU_n + U_n - U_{n-1} = 0 \qquad n = 1, \dots, N$$

Multiply by U_n:

$$akU_n^2 + U_n^2 - U_{n-1}U_n = 0 \Rightarrow U_n^2 - U_{n-1}U_n = -akU_n^2 \le 0$$

$$\Rightarrow 0 \geq \sum_{n=1}^{N} (U_n^2 - U_{n-1}U_n) = \sum_{n=1}^{N} (U_n^2 - U_{n-1}U_n) + \frac{1}{2}U_0^2 - \frac{1}{2}u_0^2 =$$

$$= \frac{1}{2}\sum_{n=1}^{N} U_n^2 + \frac{1}{2}\sum_{n=1}^{N} (U_n^2 - 2U_{n-1}U_n) + \frac{1}{2}U_0^2 - \frac{1}{2}u_0^2 =$$

$$= \frac{1}{2}U_N^2 + \frac{1}{2}\sum_{n=1}^{N} (U_n^2 - 2U_{n-1}U_n + U_{n-1}^2) - \frac{1}{2}u_0^2 =$$

$$= \frac{1}{2}U_N^2 + \frac{1}{2}\sum_{n=1}^{N} (U_n - U_{n-1})^2 - \frac{1}{2}u_0^2 =$$

$$= \frac{1}{2}U_N^2 + \frac{1}{2}\sum_{n=1}^{N} [U_{n-1}]^2 - \frac{1}{2}u_0^2$$

$$\Rightarrow \frac{1}{2}|U_N|^2 + \frac{1}{2}\sum_{n=1}^{N} |[U_{n-1}]|^2 \leq \frac{1}{2}|u_0|^2$$

$$\Rightarrow |U_N|^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \leq |u_0|^2$$

//.

20) (Belongs to w.4)

Consider the BVP

$$\begin{cases} -\varepsilon u'' + xu' + u = f & \text{in } I = (0,1), \quad \varepsilon > 0 \text{ const.} \\ u(0) = u'(1) = 0 \end{cases}$$

$f \in L^2(I)$. Prove that $\|\varepsilon u''\|_{L^2(I)} \leq \|f\|_{L^2(I)}$.

Solution Assume u satisfies BVP. Then

$$\|\varepsilon u''\|_{L^2(I)}^2 = \|xu' + u - f\|_{L^2(I)}^2 = \int_0^1 (xu' + u - f)^2\, dx =$$

$$= \underbrace{\int_0^1 (xu'+u)^2\, dx - 2\int_0^1 (xu'+u) f\, dx}_{(*)} + \underbrace{\int_0^1 f^2\, dx}_{= \|f\|_{L^2(I)}^2}$$

It is enough to show that $(*) \le 0$.

$$(*) = \left\{ f = -\varepsilon u'' + xu' + u \right\} =$$

$$= \int_0^1 (xu'+u)^2 dx + 2\varepsilon \int_0^1 (xu'+u)u'' dx - 2\int_0^1 (xu'+u)^2 dx =$$

$$= 2\varepsilon \int_0^1 (xu'+u)u'' dx - \int_0^1 (xu'+u)^2 dx =$$

$$= \left\{ \begin{array}{l} \int_0^1 (xu'+u)u'' = [(xu'+u)u']_0^1 - \int_0^1 (u'+xu''+u')u' = \\ = \left\{ \begin{array}{l} xu'+u|_0 = 0 \\ u'|_1 = 0 \end{array} \right\} = -\int_0^1 (2u'+xu'')u' = -2\int_0^1 (u')^2 - \int_0^1 xu'u'' \end{array} \right\} =$$

$$= -4\varepsilon \int_0^1 (u')^2 - 2\varepsilon \int_0^1 xu'u'' - \int_0^1 (xu'+u)^2 dx =$$

$$= \left\{ \begin{array}{l} \int_0^1 xu'u'' = [xu'u']_0^1 - \int_0^1 (u'+xu'')u' = -\int_0^1 (u')^2 - \int_0^1 xu'u'' \\ \Rightarrow 2\int_0^1 xu'u'' = -\int_0^1 (u')^2 \end{array} \right\}$$

$$= -4\varepsilon \|u'\|^2_{L^2(I)} + \varepsilon \|u'\|^2_{L^2(I)} - \|xu'+u\|^2_{L^2(I)} =$$

$$= -3\varepsilon \|u'\|^2_{L^2(I)} - \|xu'+u\|^2_{L^2(I)} \le 0 \quad \text{since } \varepsilon > 0.$$

$$\therefore \|\varepsilon u''\|^2_{L^2(I)} \le \|f\|^2_{L^2(I)}.$$

$//.$

26) If $u(x_1, x_2) = (u_1, u_2)^T$ is a vector function, then rot u is the scalar function

$$\text{rot } u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

while if u is scalar, then rot u is the vector fcn

$$\text{rot } u = \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1} \right).$$

The following identities follow directly from these definitions.

$$\text{div rot } u = 0 \qquad \text{rot grad } u = 0$$

Prove that in two dimensions and with u scalar,

$$\text{rot rot } u = -\Delta u$$

Solution

$$\text{rot rot } u = \text{rot}\left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1} \right) = \frac{\partial}{\partial x_1}\left(-\frac{\partial u}{\partial x_1} \right) - \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_2} \right) =$$

$$= -\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = -\Delta u.$$

//.

27) Prove that $-\Delta u(x) = 0, \quad x \neq 0, \quad u: \mathbb{R}^2 \to \mathbb{R}$
is given by $u(x) = \log(|x|^{-1})$.

Solution

$$\nabla u(x) = \begin{bmatrix} \dfrac{-x_1}{|x|^2} \\ \dfrac{-x_2}{|x|^2} \end{bmatrix} \qquad \Delta u(x) = \frac{x_2^2 - x_1^2}{|x|^4} + \frac{x_1^2 - x_2^2}{|x|^4} = 0$$

//.