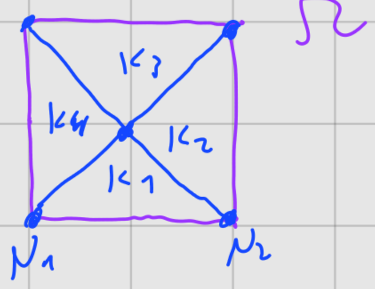
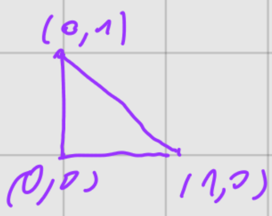
(R)

- Triangulation/mesh of $\Omega \subset \mathbb{R}^2$:



- Reference triangle $\tilde{K}$



$\lambda_1(x_1, x_2) = 1 - x_1 - x_2$

$\lambda_2(x_1, x_2) = x_1$ nodal basis of $P_1(\tilde{K})$

$\lambda_3(x_1, x_2) = x_2$

- Cont. pw linear interpolant of f: $\pi_h f \in V_h = \text{span}\left(\{\varphi_j\}_{j=1}^{n_p}\right)$ (hat fct. φ_j)

Interpolant error: $\|\pi_h f - f\|_{L^2(\Omega)} \leq C \cdot h^r \cdot \|f\|_{H^r(\Omega)}$ $\forall f \in H^r(\Omega)$ $(r=2)$

8) L^2-projection:

Def: Given $f \in L^2(\Omega)$, the L^2-projection of f is $P_h f \in V_h$ is defined by

$$\int (f - P_h f) v \, dx = 0 \qquad \forall v \in V_h \quad \text{or} \quad (f - P_h f, v) = 0 \quad \forall v \in V_h$$

$$\Omega \qquad (f - P_h f, v)_{L^2(\Omega)} = 0 \quad \forall v \in V_h$$

Rem: • The difference $f - P_h f$ is orthogonal to V_h.

• $P_h f$ can be used to approximate $f \in L^2(\Omega)$ as opposed to the interpolant $\pi_h f$, which is defined for $f \in C^0(\Omega)$ (continuous fct.)

Th: $P_h f$ is the best approximation of f in V_h in L^2-norm.

Proof:

$$\|P_h f - f\|^2_{L^2(\Omega)} \underset{\text{Def } L^2\text{-norm}}{=} (P_h f - f, P_h f - f)_{L^2(\Omega)} \underset{v \in V_h}{=} (P_h f - f, P_h f - v + v - f)_{L^2(\Omega)} \underset{\text{linearity}}{=}$$

$$= \underbrace{(P_h f - f, \underbrace{P_h f - v}_{\in V_h})_{L^2(\Omega)}}_{= 0 \text{ by orthogonality}} + (P_h f - f, v - f)_{L^2(\Omega)}$$

$$\underset{C\text{-}S}{\leq} \|P_h f - f\|_{L^2(\Omega)} \cdot \|v - f\|_{L^2(\Omega)}$$

$$\Rightarrow \quad \|P_h f - f\|_{L^2(\Omega)} \leq \|v - f\|_{L^2(\Omega)} \quad \forall v \in V_h \qquad :-) \quad \blacksquare$$

Using the above result, provide us with the error estimate for L^2-proj.

Th: Under some technical assumptions, one has

$$\|P_h f - f\| \leq \|\pi_h f - f\|_2 \leq C h^2 \|f\|$$

$$\|P_h f - f\|_{L^2(\Omega)} \leq \|\pi_h f - f\|_{L^2(\Omega)} \leq C h^2 \|f\|_{H^2(\Omega)}$$

previous section

$$\text{Proof: } \|P_h f - f\|_{L^2(\Omega)} \underset{\substack{\uparrow\\ L^2\text{-projection = best approx. in } V_h \\ \pi_h f \in V_h}}{\leq} \|\pi_h f - f\|_{L^2(\Omega)} \overset{b}{\leq} C \cdot h^2 \cdot \|f\|_{H^2(\Omega)}$$

$\blacksquare$

Chapter X: FEM for Poisson's equation

Goal: Derive FEM for Poisson and give error estimates for FEM.

Recall,

Poisson eq. (P) $\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega \end{cases}$

The variational formulation of (P) reads

(VF) Find $u \in H_0^1$ s.t. $\displaystyle\int_\Omega \nabla u \cdot \nabla v \, d\vec{x} = \int_\Omega f v \, d\vec{x} \quad \forall v \in H_0^1$,

$d\vec{x}$: dxdy

$\int_\Omega$: $\iint_\Omega$ double integrals!

where $H_0^1 = H_0^1(\Omega) = \{ v \in H^1(\Omega) \text{ s.t. } v|_{\partial\Omega} = 0 \}$

1) FE approximation:

Let T_h be a triangulation of $\Omega \subset \mathbb{R}^2$ and V_h the space of cont. pw linear fct on T_h.

Define $V_h^0 = \{ v \in V_h : v|_{\partial\Omega} = 0 \}$ (homog. Dirichlet BC)

The FE prob the reads

(FE) Find $u_h \in V_h^0$ s.t. $\displaystyle\int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx \quad \forall v_h \in V_h^0$

In order to find the FE approx. u_h, we derive a linear syst. of equations for (FE).

Observe that $\overset{\circ}{V_h} = \mathrm{span}\left(\{\varphi_j\}_{j=1}^{n_i}\right)$, where n_i is the number of interior nodes (no nodes on boundary)

Write then $u_h = \sum_{j=1}^{n_i} \xi_j \varphi_j$ and take $v_h = \varphi_i$ for $i = 1, \dots, n_i$ into (FE) to get

$$\sum_{j=1}^{n_i} \xi_j (\nabla\varphi_j, \nabla\varphi_i)_{L^2(\Omega)} = (f, \varphi_i)_{L^2(\Omega)} \quad \text{for } i = 1, \dots, n_i$$

That is the linear syst of equations

$$S \cdot \xi = b$$

, where $S \rightarrow$ stiffness matrix $(n_i \times n_i)$

$b \rightarrow$ load vector $(n_i \times 1)$

$\xi \rightarrow$ unknown $(n_i \times 1)$

Solving the above gives ξ and then

$u_h = \sum_{j=1}^{n_i} \xi_j \varphi_j$ the FE approx. of u sol. (P)

$(c\,Gr(1))$

Notation $\int_\Omega g(x)\,dx = \iint_\Omega g(x_1, x_2)\,dx_1\,dx_2 = \iint_\Omega g(x,y)\,dx\,dy$

2) Notes on implemention:

a) Data structures for a triangulation:

Standard way to represent a triangular mesh on a computer is to use 2 matrices.

Mesh T_h, $n_p \to$ number of nodes in T_h

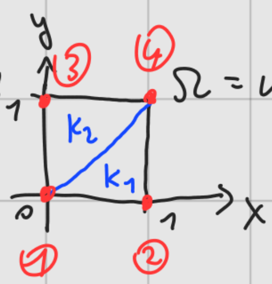
$n_t \to$ number of elements in T_h

These 2 matrices are called

the <u>points matrix P</u>

the <u>connectivity matrix T</u>

Ex:



Ω = unit square

The matrix P is given

$$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{array}{l} \leftarrow \text{coord. of } 1 \\ \\ \\ \leftarrow \text{coord. of node } 4 \end{array}$$

dim. $n_p \times 2$

The matrix T is given

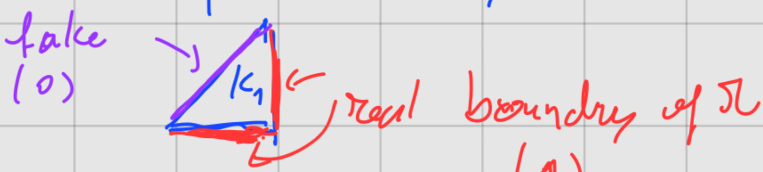
$$T = \begin{pmatrix} \overbrace{1 \quad 2 \quad 4} & \overbrace{1 \quad 1 \quad 0} \\ 1 \quad 4 \quad 3 & 0 \quad 1 \quad 1 \end{pmatrix}$$

dim: $n_t \times (3+3)$

nodes of K_1

nodes of K_2

indicates which of the triangle is a real boundary (1) or a fake boundary (0)



Rem: Can use P and T to draw mesh or refine mesh.

b) Assembly of the matrices:

Once a mesh is generated and one has P and T, one must compute the stiffness matrix S and load vector b and mass matrix, etc.

Instead of considering hat function in global manner, one decomposes the integrals and take a sum over so-called finite-element shape functions.

For the stiffness matrix S, one has:

$$\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx = \sum_{\ell=1}^{n_t} \underbrace{\int_{K_\ell} \nabla\varphi_i \cdot \nabla\varphi_j \, dx}_{=: (\nabla\varphi_i, \nabla\varphi_j)_{L^2(K_\ell)} =: \rho_{ij}^{(\ell)}}$$

In a FE code, one first computes the element stiffness matrix S_i on each triangle K_i.
The triangle K_i has nodes N_j, N_k, N_ℓ with

coordinates (x_j, y_j), resp. (x_k, y_k), resp. (x_ℓ, y_ℓ).

One has

$$S_i = \begin{pmatrix} p_{jj}^{(i)} & p_{jk}^{(i)} & p_{j\ell}^{(i)} \\ * & p_{kk}^{(i)} & p_{k\ell}^{(i)} \\ \text{Symmetric} \quad * & * & p_{\ell\ell}^{(i)} \end{pmatrix} \qquad (3 \times 3 \text{ matrix})$$

where $p_{jk}^{(i)} = (\nabla\varphi_j, \nabla\varphi_k)_{L^2(K_i)}$

Next, one adds this element contribution S_i at the appropriate location j, k, ℓ in the global matrix S:

$$\underset{\substack{\uparrow \\ \text{size} \\ n_p \times n_p}}{S} = S + \begin{array}{c} \begin{pmatrix} 0 & p_{jj}^{(i)} & 0 & p_{jk}^{(i)} & 0 & p_{j\ell}^{(i)} \\ 0 & 0 & 0 & 0 & & 0 \\ 0 & p_{jk}^{(i)} & 0 & p_{kk}^{(i)} & 0 & p_{k\ell}^{(i)} \\ 0 & p_{j\ell}^{(i)} & 0 & p_{k\ell}^{(i)} & 0 & p_{\ell\ell}^{(i)} \end{pmatrix} \begin{array}{c} j \\ \\ k \\ \ell \end{array} \\ \begin{array}{cccccc} \quad & j & \quad & k & \quad & \ell \end{array} \end{array}$$

This is the assembly process

This is the assembly process

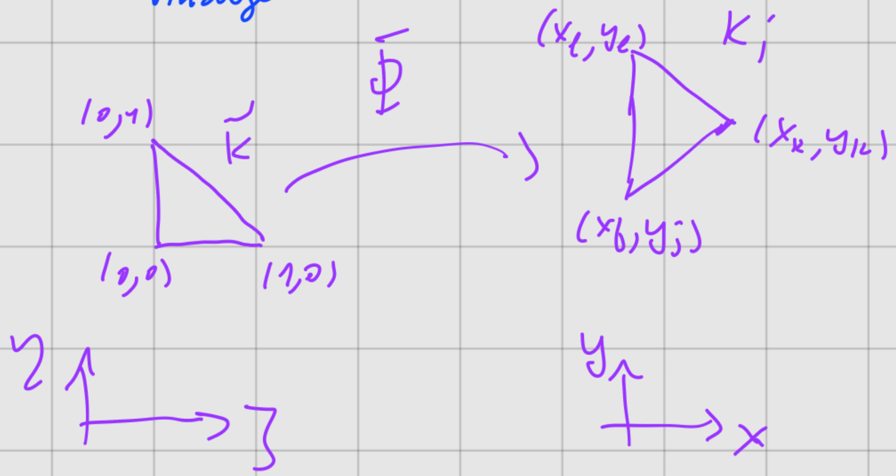
c) Computation of element matrix:

To compute the element stiffness matrix S^i, we go back to the reference triangle $\tilde{K}$.

First, introduce a linear map:

$$\Phi : \tilde{K} \longrightarrow K_i$$

reference triangle   element



$$(0,0) \longmapsto (x_j, y_j)$$
$$(1,0) \longmapsto (x_k, y_k)$$
$$(0,1) \longmapsto (x_l, y_l)$$

Next, we use the element shape fct on $\tilde{K}$:

$\tilde{\varphi}_1(\xi,\eta) = 1-\xi-\eta$

$\tilde{\varphi}_2(\xi,\eta) = \xi$

$$\tilde{\varphi}_3(\xi, \eta) = \eta$$

Observe that the linear map Φ can be defined thanks to $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3$:

$$\Phi(\xi, \eta) = \begin{pmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \end{pmatrix} \cdot \tilde{\varphi}_1(\xi,\eta) + \begin{pmatrix} x_k \\ y_k \end{pmatrix} \tilde{\varphi}_2(\xi,\eta) + \begin{pmatrix} x_\ell \\ y_\ell \end{pmatrix} \tilde{\varphi}_3(\xi,\eta).$$