(R)

- Mesh $\longrightarrow$ Point matrix and connectivity matrix on cpu.

- Assembly of stiffness matrix

$$S' = S + \begin{pmatrix} & 0 & \\ 0 & S_i & 0 \\ & 0 & \end{pmatrix}, \quad S_i \to \text{element stiffness matrix}$$

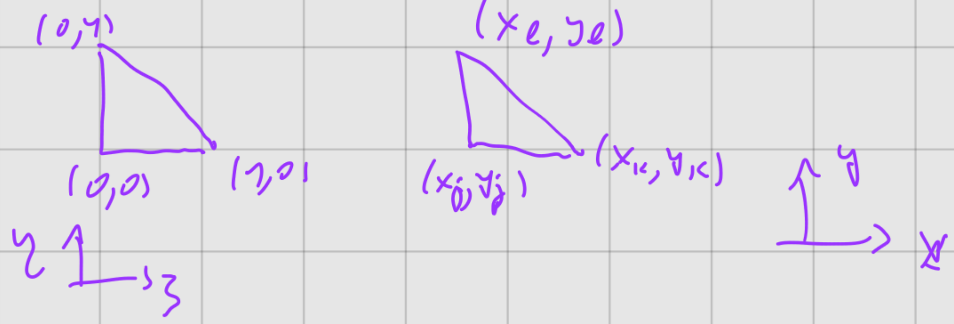
3×3

- $\Phi : \tilde{K} \longrightarrow K_i$

$$\int_{K_i} g(x,y)\,dx\,dy$$

$\Big\{$ change of coord Φ

$$\int_{\tilde{K}} \dots \, d\xi\, d\eta$$



$$\Phi(\xi,\eta) = \begin{pmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \end{pmatrix} \tilde{\varphi}_1 + \begin{pmatrix} x_k \\ y_k \end{pmatrix} \tilde{\varphi}_2 + \begin{pmatrix} x_e \\ y_e \end{pmatrix} \tilde{\varphi}_3$$

Shape fct on $\tilde{K}$:
$$\tilde{\varphi}_1(\xi,\eta) = 1 - \xi - \eta$$
$$\tilde{\varphi}_2(\xi,\eta) = \xi$$
$$\tilde{\varphi}_3(\xi,\eta) = \eta$$

Using the definition of the shape functions $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3$, one can rewrite the linear map Φ as:

$$\Phi(\xi,\eta) = \begin{pmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \end{pmatrix} + \begin{pmatrix} x_k - x_j & x_l - x_j \\ y_k - y_j & y_l - y_j \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

With all these preparations, an integration over the element K_i can be performed on the reference triangle $\tilde{K}$:

$$\int_{K_i} g(x,y)\,dx\,dy = \int_{\tilde{K}} g(\Phi(\xi,\eta))\,|J(\xi,\eta)|\,d\xi\,d\eta \,,$$

change of variable using Φ

where $|J(\xi,\eta)|$ is the determinant of the Jacobian of the linear map $\Phi(\xi,\eta)$.

$$\text{1d:} \quad \int g(u)\,du \underset{du = \Phi'(x)dx}{\overset{u=\Phi(x)}{=}} \int g(\Phi(x))\,\Phi'(x)\,dx$$

The matrix $J(\xi,\eta)$ is given by

$$J(\xi,\eta) = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \cdots \end{pmatrix} = \begin{pmatrix} x_k - x_j & x_l - x_j \\ \cdots \end{pmatrix}$$

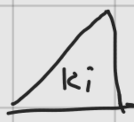
$$\begin{pmatrix} \cdots & \cdots \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} \quad \begin{pmatrix} \cdots & \cdots \\ y_k - y_j & y_l - y_j \end{pmatrix}$$

Def

$$\Phi(\xi,\eta) = \begin{pmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{pmatrix}$$

Using the def of the linear map Φ seen above

Rem: • The 2×2 matrix $J(\xi,\eta)$ only depends on the nodes of the element K_i.

• Area () $= \frac{1}{2}$ Area () = use

cross prod. to compute this area =

$= \frac{1}{2} \det(J(\xi,\eta)) \Rightarrow |J(\xi,\eta)| = \det(J(\xi,\eta)) =$

$= 2 \cdot \text{Area}(K_i) \Rightarrow$

Illustration:

For the stiffness matrix, one needs to compute

$\int_{K_i} \nabla\varphi_j \cdot \nabla\varphi_k \, dx\, dy$. We now provide details

how to do this:

$$\int_{K_i} \nabla \varphi_j \cdot \nabla \varphi_k \, dx\,dy \underset{\text{def inner prod.}}{=} \int_{K_i} \nabla \varphi_j^T \nabla \varphi_k \, dx\,dy \underset{\substack{\text{change of variable } K_i \leftrightarrow \tilde{K} \\ + \text{ chain rule} \\ (*)}}{=}$$

$(*)$ $\tilde{\varphi} = \varphi \circ \Phi \rightarrow \nabla \tilde{\varphi}^T = \nabla \varphi^T J$ or $\nabla \varphi^T = \nabla \tilde{\varphi}^T J^{-1}$

and $\nabla \varphi = J^{-T} \nabla \tilde{\varphi}$

$$= \int_{\tilde{K}} \left(\nabla \tilde{\varphi}_1^T J^{-1}\right)\left(J^{-T} \nabla \tilde{\varphi}_2\right) |J(\xi,\eta)| \, d\xi\, d\eta =$$

$$\underset{\uparrow}{=} (-1,-1)\, J^{-1} J^{-T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} |J(\xi,\eta)| \cdot \int_{\tilde{K}} d\xi\, d\eta =$$

Recall $\tilde{\varphi}_1(\xi,\eta) = 1-\xi-\eta \rightarrow \nabla \tilde{\varphi}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ independent

$\tilde{\varphi}_2(\xi,\eta) = \xi \rightarrow \nabla \tilde{\varphi}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of (ξ,η)

$$= (-1,-1)\, J^{-1} J^{-T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \underbrace{\cdot 2 \cdot \text{Area}(K_i)}_{|J|} \cdot \underbrace{\frac{1}{2}}_{\text{area}(\tilde{K})}$$

Summary: The integral, $\int_{K_i} \nabla \varphi_j \nabla \varphi_k$, can be

computed using the above formula, which only depends on the nodes of the element K_i.

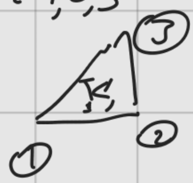
This information is provided by the matrices P, T.

d) Computation of the load vector:

The load vector b is assembled using the same technique as for the stiffness matrix S:
On each element K_i, one gets a local 3×1 element load vector b_{K_i} with entries

K_i element

$$(b_{K_i})_i = \int_{K_i} f(x,y)\,\varphi_i(x,y)\,dx\,dy \quad \text{for } i = 1,2,3$$



An approximation of the above integral is given by:

$$\int_{K_i} f(x,y)\,\varphi_i(x,y)\,dx\,dy \approx f(N_i)\cdot \int_{K_i} \varphi_i(x,y)\,dx\,dy =$$

$f(x,y) \approx f(N_i)$ where N_i denotes nodes i.

$$= f(N_i)\cdot \int_{\tilde{K}} \tilde{\varphi}_i(\xi,\eta)\,\overbrace{|J(\xi,\eta)|}^{2\cdot \text{Area}(K)}\,d\xi\,d\eta$$

$2\cdot\text{Area}(K)$ since the matrix J depends on the linear map Φ that depends only on K_i.

change of variable $\Phi : \tilde{K} \to K$ (element), J = Jacobian of Φ

$\tilde{K}$: reference triangle

$$= f(N_i) \cdot 2 \cdot \text{Area}(K_i) \cdot \underbrace{\int_{\tilde{K}} \varphi_i(\xi,\eta)\, d\xi\, d\eta}_{\frac{1}{6} \text{ for } i=1,2,3}$$

$$= f(N_i) \cdot \frac{1}{3} \text{Area}(K)$$

$$\Longrightarrow \quad b_{K_i} \approx \frac{1}{3} \text{Area}(K_i) \begin{pmatrix} f(N_1) \\ f(N_2) \\ f(N_3) \end{pmatrix}$$, where N_1, N_2, N_3 are the nodes of element K_i.

3) A priori error estimates:

We start with

Th: (Poincaré ineq.) Let $\Omega \subset \mathbb{R}^2$ bounded domain with smooth boundary / polygonal $\partial\Omega$. Then, $\exists$ constant $C>0$ s.t.

$$\|v\|_{L^2(\Omega)} \le C \cdot \|\nabla v\|_{L^2(\Omega)} \quad \forall\, v \in H_0^1(\Omega).$$

Proof: • Let Φ be s.t. $-\Delta\Phi = 1$ in Ω and

$$\sup_{x\in\Omega} |\nabla\Phi(x)| \le C \qquad (\text{see the end of proof})$$

• Next, one has

$1 = -\Delta\Phi$

$$\|v\|^2_{L^2(\Omega)} \overset{\text{Def norm}}{=} \int_\Omega v^2 \, dx_1 dx_2 = \int_\Omega v^2 \cdot 1 \, dx_1 dx_2 =$$

$x = (x_1, x_2)$

$$= -\int_\Omega v^2 \Delta\Phi \, dx_1 dx_2 \overset{\text{Green}}{=} -\underbrace{\int_{\partial\Omega} v^2 (n\cdot\nabla\Phi)\, ds}_{=0 \text{ since } v\in H_0^1} +$$

$$+ \int_\Omega 2v \nabla v \cdot \underbrace{\nabla\Phi}_{\sup_{x\in\Omega} |\nabla\Phi(x)| \leq C \text{ by def } \Phi} \, dx_1 dx_2$$

$$\leq C \cdot \int_\Omega v |\nabla v| \, dx_1 dx_2 \overset{C\text{-}S}{\leq} C \cdot \|v\|_{L^2(\Omega)} \cdot \|\nabla v\|_{L^2(\Omega)}$$

That is $\|v\|_{L^2(\Omega)} \leq C \cdot \|\nabla v\|_{L^2(\Omega)}$:-)

- Expl. of such Φ is given by: $\Phi(x_1, x_2) = -\frac{1}{4}(x_1^2 + x_2^2)$.

Indeed, $\nabla\Phi(x_1,x_2) = \begin{pmatrix} -1/2\, x_1 \\ -1/2\, x_2 \end{pmatrix} \to |\nabla\Phi(x_1,x_2)| \overset{\text{Euclid. norm}}{=} \sqrt{\frac{1}{4}x_1^2 + \frac{1}{4}x_2^2} =$

$$= \frac{1}{2}\sqrt{x_1^2 + x_2^2} = \frac{1}{2}\,\mathrm{Diam}(\Omega) \leq$$

$$\leq C \text{, where } C \text{ only dep. on } \Omega.$$

$$\Delta\Phi(x_1,x_2) = -\frac{1}{2} - \frac{1}{2} = -1$$

Next, we prove

Th: (Galerkin orthogonality)

Let u, resp. u_h be solutions to (VF), resp. (FE). Assume that u, u_h are nice enough. One has GO:

double integral

$$(GO) \quad \int_\Omega \nabla(u-u_h)\cdot\nabla v_h \, dx_1 dx_2 = 0 \qquad \forall v_h \in V_h^0$$

[$u-u_h$ $\perp$ to V_h^0 in energy inner prod.]

[$-u'' = f$ $\rightarrow$ $\|\cdot\|_E$ and $(\cdot,\cdot)_E$

$a(u,v) = \dots$ $\rightarrow$ $\|u\|_E = \sqrt{a(u,u)}$]

Proof:

Recall that VF and FE are given by

$$(VF) \dots \quad (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1$$

$$(FE) \dots \quad (\nabla u_h, \nabla v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h^0 \subset H_0^1$$

Hence, one can take test fct v_h in (VF)!
Substract (FE) from (VF) gives:

$$\underbrace{(\nabla u, \nabla v_h)_{L^2} - (\nabla u_h, \nabla v_h)_{L^2}}_{(\nabla(u-u_h), \nabla v_h)_{L^2}} = (f, v_h)_{L^2} - (f, v_h)_{L^2} = 0$$

using linearity

[illegible]