34) If $-\Delta E = \delta_0$ in $\mathbb{R}^d$, where δ_0 is the Dirac Delta distribution, then E is called the fundamental solution of $-\Delta$ in $\mathbb{R}^d$. Equivalently, for any smooth test function v vanishing outside of a bounded set, i.e. for any $v \in C_0^\infty(\mathbb{R}^d)$, E satisfies:

$$-\int_{\mathbb{R}^d} \Delta E(x) v(x) = v(0) \qquad \left(\int \delta_0 v \overset{\text{def}}{=} v(0) \right)$$

Prove that $E(x) \overset{(*)}{=} \frac{1}{2\pi} \log\left(\frac{1}{|x|}\right)$ is a fundamental solution of $-\Delta$ in $\mathbb{R}^2$.

Solution

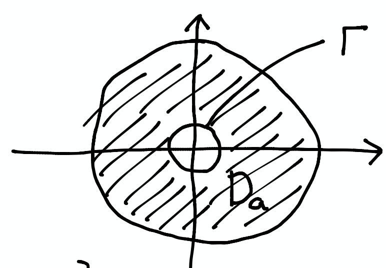
We want to show that

$$-\int_{\mathbb{R}^2} \Delta\left(\frac{1}{2\pi} \log\left(\frac{1}{|x|}\right)\right) v(x) = v(0) \quad \forall\, v \in C_0^\infty$$

In order to apply Green's, we need a C^2-function so we need to remove the problematic part around $x=0$.

Take $v(x) \in C_0^\infty$. Let $D_a := \{x \in \mathbb{R}^2 : 0 < a < |x| < \frac{1}{a}\}$, with a small enough so that $\operatorname{supp} v \subset \{x \in \mathbb{R}^2 : |x| < \frac{1}{a}\}$, i.e. v vanishes outside of $|x| = \frac{1}{a}$.

Let $\Gamma := \{x \in \mathbb{R}^2 : |x| = a\}$



We then get

$$-\int_{D_a} \Delta E v = \left\{ \begin{array}{l} \text{Green's formula} \\ \int_\Omega \Delta u v = \int_{\partial\Omega} (\nabla u \cdot n) v - \int_\Omega \nabla u \cdot \nabla v \end{array} \right\} =$$

$$= -\int_{\partial D_a} (\nabla E \cdot n) v + \int_{D_a} \nabla E \cdot \nabla v = \{\text{Green's}\} =$$

$$= -\int_{\partial D_a} (\nabla E \cdot n) v + \int_{\partial D_a} (\nabla v \cdot n) E - \int_{D_a} E \Delta v = \left\{ \begin{array}{l} v \text{ vanishes} \\ \text{before outer} \\ \text{boundary } |x| = \frac{1}{a} \end{array} \right\} =$$

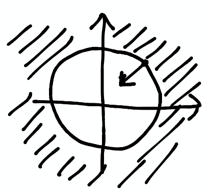
$$= -\int_{\Gamma} (\nabla E \cdot n) v + \int_{\Gamma} (\nabla v \cdot n) E - \int_{D_a} E \Delta v .$$

Exercise 27: $-\Delta E = 0, \ x \neq 0$ with E defined as $(*)$.

$$\Rightarrow \ -\int_{D_a} \Delta E \, v = 0$$

$$\Rightarrow \ \int_{D_a} E \Delta v = \int_{\Gamma} (\nabla v \cdot n) E - \int_{\Gamma} \underbrace{(\nabla E \cdot n)} v$$

What is $\nabla E \cdot n$ on Γ?



$|x| = a$

$|n| = 1$

$\Rightarrow n = \frac{-x}{a}$

$$\nabla E = -\frac{1}{2\pi} \frac{x}{|x|^2}$$

$$\Rightarrow \nabla E \cdot n = -\frac{1}{2\pi} \cdot \frac{x}{|x|^2} \cdot \frac{-x}{a} = \frac{1}{2\pi a}$$

$$\Rightarrow \int_{D_a} E \Delta v = \int_{\Gamma} \frac{1}{2\pi} \ln\left(\frac{1}{|x|}\right) \nabla v \cdot n - \int_{\Gamma} \frac{1}{2\pi a} v =$$

$$= \frac{1}{2\pi} \ln\left(\frac{1}{a}\right) \underbrace{\int_{\Gamma} \nabla v \cdot n}_{\substack{v \in C^\infty(\mathbb{R}^2) \\ \Gamma \text{ smooth} \\ \Rightarrow \int_\Gamma \nabla v \cdot n = \nabla v \cdot n|_\eta \int_\Gamma ds \\ \text{some } \eta \in \Gamma}} - \frac{1}{2\pi a} \underbrace{\int_{\Gamma} v}_{\substack{v \in C^\infty(\mathbb{R}^2) \\ \Gamma \text{ smooth} \\ \Rightarrow \exists \xi \in \Gamma: \\ \int_\Gamma v = v(\xi) \int_\Gamma ds}} = \left\{ \int_{\Gamma} ds = 2\pi a \right\}$$

$$= \frac{2\pi a}{2\pi} \ln\left(\frac{1}{a}\right) (\nabla v \cdot n)_\eta - \frac{2\pi a}{2\pi a} v(\xi) =$$

$$= \underbrace{a\ln\left(\tfrac{1}{a}\right)}_{\to 0,\ a\to 0^+} \underbrace{(\nabla v\cdot n)_\eta}_{\text{bounded}} - v(\xi) \longrightarrow -v(0), \quad a \to 0^+$$

$$\Rightarrow -\int_{\mathbb{R}^2} \Delta E(x)\, v(x) \overset{def}{=} -\int_{\mathbb{R}^2} E(x)\, \Delta v(x) = v(0) \quad . \text{ any } v \in C_0^\infty.$$

40) Consider the Poisson equation with non-homog. Dirichlet boundary conditions

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega \\ u(x) = g(x), & x \in \Gamma = \partial\Omega \end{cases}$$

where $g(x)$ is given boundary data. The variational formulation takes the following form:

find $u \in V_g$, where $V_g := \{v : v = g \text{ on } \Gamma,\ \int_\Omega (|\nabla v|^2 + v^2)\,dx < \infty\}$

such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall\ v \in V_0$$

where $V_0 := \{v : v = 0 \text{ on } \Gamma,\ \int_\Omega (|\nabla v|^2 + v^2)\,dx < \infty\}$

* Show that V_g is not a vector space.
* Prove that the solution of the weak form is unique.

Solution

* Let $g(x) \not\equiv 0$ on Γ. Let $u, v \in V_g$. $u+v|_\Gamma = 2g(x)$
  $\Rightarrow u+v \notin V_g$. Hence not a vector space.

* To show uniqueness, assume $u_1, u_2 \in V_g$ such that $(\nabla u_1, \nabla v) = (f, v)$ and $(\nabla u_2, \nabla v) = (f, v)$ $\forall v \in V_0$.

$\Rightarrow w = u_1 - u_2 \in V_0$, satisfies $(\nabla w, \nabla v) = 0$ $\forall v \in V_0$.

Let $v = w$. Then $\|\nabla w\|^2 = 0$

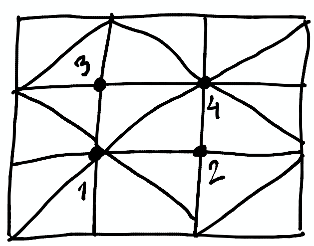
By Poincaré, $0 \le \|w\|_{L^2}^2 \le C\|\nabla w\|_{L^2}^2 = 0$, $w \in V_0 = H_0^1$

$\|\cdot\|_{L^2}$ = norm $\Rightarrow w \equiv 0$ a.e. $\Rightarrow u_1 = u_2$ a.e. %

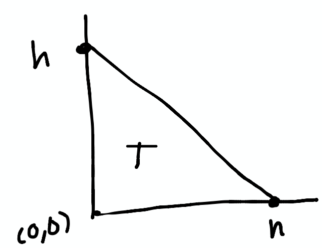
9.12) Formulate the cG(1) method for the BVP

$$\begin{cases} -\Delta u + u = f, & x \in \Omega \subset \mathbb{R}^2 \\ u = 0, & x \in \partial\Omega \end{cases}$$

Write down the matrix form of the resulting system of equations using the uniform mesh:



standard element



Solution

VF: $-\int_\Omega \Delta u v + \int_\Omega uv = \int_\Omega fv \quad \forall v \in H_0^1(\Omega)$

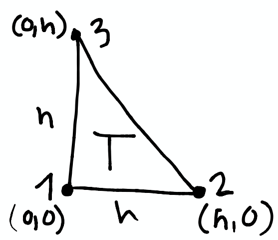
Green's + $\int_{\partial\Omega} (\nabla u \cdot n) v = 0$, $v \in H_0^1$

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv = \int_\Omega fv \quad \forall v \in H_0^1$$

<u>FEM</u>: Let $u_h = \sum_{j=1}^{4} \xi_j \varphi_j$, φ_j = tent functions with $\varphi_j = 1$ in the point j.

$$\Rightarrow \sum_{j=1}^{4} \xi_j \Big(\underbrace{\int_\Omega \nabla\varphi_j \nabla\varphi_i}_{s_{ij}} + \underbrace{\int_\Omega \varphi_j \varphi_i}_{m_{ij}} \Big) = \int_\Omega f \varphi_i \,, \quad i = 1, \dots, 4$$

For the standard element T, we have:



$\phi_1(x,y) = (h-x-y)/h$ $\qquad \nabla\phi_1 = \left(-\frac{1}{h}, -\frac{1}{h}\right)$

$\phi_2(x,y) = x/h$ $\qquad \nabla\phi_2 = \left(\frac{1}{h}, 0\right)$

$\phi_3(x,y) = y/h$ $\qquad \nabla\phi_3 = \left(0, \frac{1}{h}\right)$

area : $h^2/2$

<u>Find local stiffness matrix, S</u>

$$\int_T \nabla\phi_1 \cdot \nabla\phi_1 = \int_T \frac{2}{h^2} = \frac{h^2}{2} \cdot \frac{2}{h^2} = 1$$

$$\int_T \nabla\phi_1 \cdot \nabla\phi_2 = \int_T \nabla\phi_1 \cdot \nabla\phi_3 = \dots = -\frac{1}{2}$$

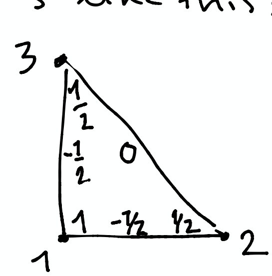
$$\int_T \nabla\phi_2 \cdot \nabla\phi_2 = \int_T \nabla\phi_3 \cdot \nabla\phi_3 = \dots = \frac{1}{2}$$

$$\int_T \nabla\phi_2 \cdot \nabla\phi_3 = 0$$

$$S = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

when assembling the global stiffness matrix, it may be easier to view S like this:

S =



Find local mass matrix, m:

$$\int_T \phi_1\phi_1 = \ldots = \frac{h^2}{12}$$

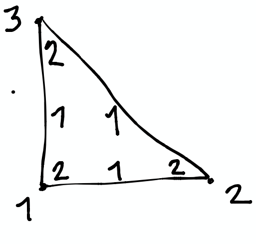
$$\int_T \phi_1\phi_2 = \int_T \phi_1\phi_3 = \frac{h^2}{24}$$

$$\int_T \phi_2\phi_2 = \int_T \phi_3\phi_3 = \frac{h^2}{12}$$

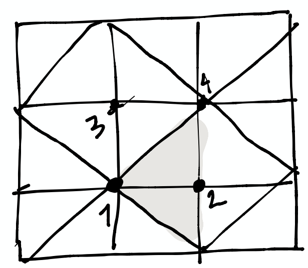
$$\int_T \phi_2\phi_3 = \ldots = \frac{h^2}{24}$$

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$m = \frac{h^2}{24} \cdot$$



Assemble the global matrices:



$$S = \begin{bmatrix} 8\cdot s_{22} & 2\cdot s_{12} & 2\cdot s_{12} & 2\cdot s_{23} \\ 2\cdot s_{12} & 4\cdot s_{11} & 0 & 2\cdot s_{12} \\ 2\cdot s_{12} & 0 & 4\cdot s_{11} & 2\cdot s_{12} \\ 2\cdot s_{23} & 2\cdot s_{12} & 2\cdot s_{12} & 8\cdot s_{22} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

$$M = \begin{bmatrix} 8m_{22} & 2\cdot m_{12} & 2\cdot m_{12} & 2m_{23} \\ 2\cdot m_{12} & 4\cdot m_{11} & 0 & 2m_{12} \\ 2\cdot m_{12} & 0 & 4\cdot m_{11} & 2\cdot m_{12} \\ 2\cdot m_{23} & 2m_{12} & 2m_{12} & 8m_{22} \end{bmatrix} = \frac{h^2}{24}\begin{bmatrix} 16 & 2 & 2 & 2 \\ 2 & 8 & 0 & 2 \\ 2 & 0 & 8 & 2 \\ 2 & 2 & 2 & 16 \end{bmatrix}$$

Resulting system:

$$(S+M)\xi = b, \quad b = \left[\int_\Omega f\varphi_1 \ \int_\Omega f\varphi_2 \ \int_\Omega f\varphi_3 \ \int_\Omega f\varphi_4\right]^T.$$

39) For the linear system $M\eta = b$, where M is the mass matrix w/ coeff. $(\varphi_j, \varphi_i)_{i,j=1}^{m}$, b is load vector (v, φ_i), prove that M is symmetric, pos. def.

Solution
Symmetry obvious.

Positive definite

For $v \in \mathbb{R}^m \setminus \{0\}$:

$$v^T M v = \sum_{i=1}^{m} \sum_{j=1}^{m} v_i (\varphi_j, \varphi_i) v_j = \left(\sum_{j=1}^{m} v_j \varphi_j , \sum_{i=1}^{m} v_i \varphi_i \right) =$$

$$= \|w\|_{L^2}^2 > 0 \quad \text{since } w := \sum_{i=1}^{m} v_i \varphi_i \neq 0 \text{ since } v \neq 0.$$

$\%$

37) Describe a discrete system of equations for a p.w. polynomial approximation for

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

with continuous polynomial basis functions, $\{\varphi_j\}_{j=1}^{M}$. Show that the stiffness matrix for this approximation is symmetric and positive definite.

Solution

Let $V_h := \{v : \text{p.w. polynomial, continuous, } v \equiv 0 \text{ on } \Gamma\} =$
$= \text{span}\{\varphi_j\}_{j=1}^{M}$

Discrete VF of the problem: Find $u_h \in V_h$ s.t $(\nabla u_h, \nabla v) = (f, v)$ $\forall v \in V_h$. Let $u_h = \sum_{j=1}^{M} \xi_j \varphi_j$. Then the problem is equivalent to finding $\xi = (\xi_j)_{j=1}^{M} \in \mathbb{R}^M$ such that

$\sum_{j=1}^{M} \xi_j (\nabla \varphi_j, \nabla \varphi_i) = (f, \varphi_i)$ for $i = 1, \dots, M$

This can be written $A\xi = b$, $A = (a_{ij})_{i,j=1}^{M} = ((\nabla\varphi_j, \nabla\varphi_i))_{i,j=1}^{M}$

$b = (b_i)_{i=1}^{M} = ((f, \varphi_i))_{i=1}^{M}$

A is obviously symmetric. Positive definite? ($v^T A v > 0$ for every $v \in \mathbb{R}^M \setminus \{0\}$.)

Let $v \in \mathbb{R}^M \setminus \{0\}$ and let $w = \sum_{i=1}^{M} v_i \varphi_i \in V_h \setminus \{0\}$

$$v^T A v = \sum_{i=1}^{M} \sum_{j=1}^{M} v_i (\nabla\varphi_j, \nabla\varphi_i) v_j = \left(\sum_{j=1}^{M} v_j \nabla\varphi_j, \sum_{i=1}^{M} v_i \nabla\varphi_i \right) =$$

$$= \left(\nabla \sum_{j=1}^{M} v_j \varphi_j, \nabla \sum_{i=1}^{M} v_i \varphi_i \right) = (\nabla w, \nabla w) = \|\nabla w\|_{L^2}^2 \geq 0$$

$v^T A v = 0 \iff \nabla w = 0 \iff w$ constant $\iff w \equiv 0$ in Ω
$\iff v = 0$.

since $w = 0$ on $\partial\Omega$ and continuous.

Thus $v^T A v > 0$ for $v \neq 0$ and A is positive definite.

//.

38) We define the L^2-projection $P_h v$ of a function $v \in L^2(\Omega)$ into the finite element space by

$$(P_h v, w) = (v, w) \quad \forall w \in V_h.$$

We also define the discrete Laplacian Δ_h by

$$-(\Delta_h w, v) = (\nabla w, \nabla v) \quad \forall v \in V_h.$$

Verify that we may express the finite problem

$$(\nabla U, \nabla v) = (f, v) \quad \forall v \in V_h$$

as finding $U \in V_h$ such that

$$-\Delta_h U = P_h f.$$

Solution

$$-\Delta_h U = P_h f \quad \text{in } V_h \quad \text{(Uniqueness by L-M)}$$

$$\Longleftrightarrow \quad -(\Delta_h U, v) = (P_h f, v) \quad \forall v \in V_h$$

$$\overset{\text{def } \Delta_h}{\Longleftrightarrow} \quad (\nabla U, \nabla v) = (P_h f, v) \quad \forall v \in V_h$$

$$\overset{\text{def } P_h}{\Longleftrightarrow} \quad (\nabla U, \nabla v) = (f, v) \quad \forall v \in V_h$$

%.