R • PDE $\longleftrightarrow$ ODE in Hilbert space

Ex: Heat eq.
$$\begin{cases} \dot u(t) = \Delta u(t) + f(t) \\ u(0) = u_0 \end{cases}$$

• Duhamel's formula / variation of constant formula

$$u(t) = E(t)u_0 + \int_0^t E(t-s) f(s)\,ds$$

• VF for heat eq.

$$(H) \begin{cases} u_t - \Delta u = f & \text{in } \Omega \subset \mathbb{R}^n, \text{ for } 0 < t \le T \\ u = 0 & \text{on } \partial\Omega, \text{ for } 0 < t \le T \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases}$$

, where $u = u(x,t)$,

reads

(VF) For each $0 < t \le T$, find $u(\cdot, t) \in H_0^1(\Omega)$ s.t.

$$(u_t(\cdot,t), v)_{L^2} + a(u(\cdot,t), v) = (f(\cdot,t), v)_{L^2} \quad \forall v \in H_0^1$$

$$u(x,0) = u_0(x),$$

where bilinear form $a(u,v) = (\nabla u, \nabla v)_{L^2}$

• Exam[illegible] simple [illegible] (1, 5, 7?)

• Exam: simple pocket calculator ([illegible]/4 + [illegible]/6 − [illegible]/8)

• Last lecture on 04.03.22?

(i) VF for heat eq.

(ii) To derive the FE problem corresponding to (VF), we consider a triangulation/mesh of Ω denoted by T_h and the space V_h of cont. pw. linear fct. on T_h. Define $V_h^0 = \{ v \in V_h : v_{|\partial\Omega} = 0 \} = \text{span}\left(\{\varphi_j\}_{j=1}^{n_i} \right)$, where n_i = nbr of internal nodes.

One then has the FE prob:

(FE) For each $0 < t \leq T$, find $u_h(\cdot, t) \in V_h^0$ s.t.

$$(u_{h,t}(\cdot,t), v_h)_{L^2} + a(u_h(\cdot,t), v_h) = (f(\cdot,t), v_h) \quad \forall v_h \in V_h^0$$

$$u_h(\cdot, t) = \Pi_h u_0(\cdot)$$

↳ cont. pw linear interpolant of u_0.

(iii) To get a linear syst. of ODE from (FE), we write

$$u_h(x,t) = \sum_{j=1}^{n_i} \xi_j(t) \varphi_j(x)$$

and take $v_h(x) = \varphi_i(x)$, for $i = 1, 2, \dots, n_i$.

As in 1d, one obtains the ODE:

$$\begin{cases} \mathcal{M}\,\dot{\xi}(t) + \mathcal{S}\,\xi(t) = F(t) \\ \xi(0) = \xi^0, \end{cases}$$

$\mathcal{M} \leftrightarrow$ mass matrix $(n_i \times n_i)$

$\mathcal{S} \leftrightarrow$ stiffness matrix $(n_i \times n_i)$

$F(t) \leftrightarrow$ vector of size $(n_i \times 1)$, entry

$$(f(\cdot, t), \varphi_i)_{L^2} \quad \text{for } i = 1, \dots, n_i.$$

$\xi^0 \leftrightarrow$ initial condition with entry

$$\Pi_h u_0(x_i) \quad \text{for } i = 1, \dots, n_i$$

$\xi(t) \leftrightarrow$ unknown

(iv) Finally, one applies f.ex implicit Euler scheme to get approximations

$$\xi^n \approx \xi(t_n) \quad , \text{ where } t_n = n \cdot k ,$$

where k stepsize

$$\Rightarrow u_h^n(x, t_n) = \sum_{j=1}^{n_i} \xi_j^n \varphi_j(x) \approx u(x, t_n).$$

3) Semi-discrete error estimate:

Recall: For the heat eq. (H), recall

- (VF) $(u_t, v)_{L^2} + a(u, v) = (f, v)$
  $u(x, 0) = u_0(x)$

Stability: $\|u(\cdot, t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|f(\cdot, s)\|_{L^2} ds$

- (FE) $(u_{h,t}, v_h)_{L^2} + a(u_h, v_h) = (f, v_h)$
  $u_h(x, 0) = \pi_h u_0(x)$

- Error of interpolation:

$$\|v - \pi_h v\|_{L^2(\Omega)} \leq C \cdot h^2 \cdot \|v\|_{H^2(\Omega)} .$$

Def: Define $R_h : H_0^1 \to V_h^0$ the orthogonal projection with respect to the energy inner product:

For $v \in H_0^1$: $a(R_h v - v, v_h) = 0 \quad \forall v_h \in V_h^0$.

the operator R_h is called Ritz projection (or elliptic operator)

Rem:
- $a(u,v) = (\nabla u, \nabla v)_{L^2}$

- Remember L^2-projection.

For the error of Ritz projection, we have:

Th: Let $\Omega \subset \mathbb{R}^2$ nice. For $s = 1,2$, one has

$$\|R_h v - v\|_{L^2(\Omega)} \leq C \cdot h^s \cdot \|v\|_{H^s(\Omega)}$$

$$\forall v \in H^s(\Omega) \cap H_0^1$$

With the above preparation, we show an error estimate for FEM when applied to heat eq. (H).

Th: Let u, resp. u_h be sol. to (VF), resp. (FE). Assume u, u_h nice enough. One has error estimate:

$$\|u_h(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)} \le \|\Pi_h u_0 - u_0\|_{L^2(\Omega)} +$$

$$+\, C h^2 \left(\|u_0\|_{H^2(\Omega)} + \int_0^t \|u_t(\cdot,s)\|_{H^2(\Omega)}\, ds \right) \le$$

$$\le C \cdot \boxed{h^2} \quad \text{for } 0<t\le T.$$

Proof:

- Write the error of FEM as

$$u-u_h = \underbrace{(u - R_h u)}_{err_1} + \underbrace{(R_h u - u_h)}_{err_2}$$

- Estimate for err_1:

$$\|u(\cdot,t) - R_h u(\cdot,t)\|_{L^2} \le C \cdot h^2 \cdot \|u(\cdot,t)\|_{H^2} \le$$

previous Theorem with $s=2$.

$$\underset{\uparrow}{\leq} C \cdot h^2 \cdot \left\| u_0 + \int_0^t u_t(\cdot, s)\, ds \right\|_{H^2} \overset{\Delta}{\leq}$$

Fund. theorem of calculus

$$\leq C \cdot h^2 \left(\|u_0\|_{H^2} + \int_0^t \|u_t(\cdot, s)\|_{H^2}\, ds \right)$$

- Error estimate for err_2:

Idea: Find an equation for err_2 ...
For $\varphi \in V_h^0$, one has

$$((err_2)_t, \varphi)_{L^2} + a(err_2, \varphi) \overset{\text{Def } err_2}{=} ((R_h u - u_h)_t, \varphi)_{L^2} +$$

$$+ a(R_h u - u_h, \varphi) \underset{\text{linearity}}{=} (R_h u_t, \varphi)_{L^2} - \underline{(u_{h,t}, \varphi)_{L^2}} +$$

$$a(R_h u, \varphi) - \underline{a(u_h, \varphi)} \underset{u_h \text{ sol. (FE)}}{=} -(f, \varphi)_{L^2} + (R_h u_t, \varphi)_{L^2}$$

$$+ \underbrace{a(R_h u, \varphi)}_{a(u,\varphi) \text{ by Def of Ritz}} = a(u, \varphi) - (f, \varphi)_{L^2} + (R_h u_t, \varphi)_{L^2} =$$

$$= -(u_t, \varphi) + (R_h u_t, \varphi) = (R_h u_t - u_t, \varphi) -$$

$$= (u_t, \varphi)_{L^2} + (R_h u_t, \varphi)_{L^2} \underset{\text{linearity}}{=} (R_h u_t - u_t, \varphi)_{L^2} =$$

u sol. (VF)

$$\underset{\text{Def } err_1.}{=} - ((err_1)_t, \varphi)_{L^2}$$

That is, we obtain a heat eq. for err_2:

$$((err_2)_t, \varphi)_{L^2} + a(err_2, \varphi)_{L^2} = -((err_1)_t, \varphi)_{L^2}$$

We can use stability estimate:

$$\| err_2(\cdot, t) \|_{L^2} \leq \| err_2(\cdot, 0) \|_{L^2} + \int_0^t \| (err_1(\cdot, s))_t \|_{L^2} \, ds$$

- It remains to bound the terms

  $\| err_2(\cdot, 0) \|_{L^2}$ and $\| (err_1(\cdot, s))_t \|_{L^2}$:

We have

$$\| err_2(\cdot, 0) \|_{L^2} \underset{\text{Def } err_2}{=} \| R_h u_0 - \Pi_h u_0 \|_{L^2} \overset{\Delta}{\leq} \| R_h u_0 - u_0 \|_{L^2} + \| u_0 - \Pi_h u_0 \|_{L^2}$$

$$\leq C \cdot h^2 \| u_0 \|_{\dots} + \| u_0 - \Pi_h u_0 \|_{L^2}$$

Error of Ritz for $s=2$
(above Theorem)

Next, we have:

$$\|(err_1(\cdot,s))_t\|_{L^2} \underset{\text{Def } err_1}{=} \|(u(\cdot,s)-R_h u(\cdot,s))_t\|_{L^2} =$$

$$= \|u_t(\cdot,s)-R_h u_t(\cdot,s)\|_{L^2} \underset{\substack{\text{error of Riesz}\\ s=2 \text{ in above Theorem.}}}{\le} C\cdot h^2\cdot \|u_t(\cdot,s)\|_{H^2}$$

$\Rightarrow$ We collect everything:

$$\|u(\cdot,t)-u_h(\cdot,t)\|_{L^2} \le \|u_0-\Pi_h u_0\|_{L^2} + Ch^2\cdot\Big(\|u_0\|_{H^2} + \int_0^t \|u_t(\cdot,s)\|_{H^2}\,ds\Big)$$

∎

Chapter XII: FEM for wave eq. in $\mathbb{R}^d$

Goal: Same programme as in previous chapter, but for wave eq.

1) Conservation of energy:

Let $\Omega \subset \mathbb{R}^d$ nice, consider homogeneous wave eq.

$$(W) \begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(\cdot,0) = u_0 & \\ u_t(\cdot,0) = v_0 & \text{in } \Omega, \end{cases}$$

$u = u(x,t)$, $x \in \Omega$, $0 \le t \le T$.

By multypling the above problem with u_t, using Green's formula, we end up with

$$\underbrace{(u_{tt}, u_t)_{L^2}}_{\frac{1}{2}\frac{d}{dt}(u_t,u_t)_{L^2(\Omega)}} + \underbrace{(\nabla u, \nabla u_t)_{L^2}}_{\frac{1}{2}\frac{d}{dt}(\nabla u,\nabla u)_{L^2(\Omega)}} = 0$$

"$\frac{d}{dt} u_t^2 \rightsquigarrow 2u_t u_{tt}$"

That is, we get, by def of $\|\cdot\|_{L^2}$, that

$$\frac{1}{2}\frac{d}{dt}\|u_t(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\nabla u(\cdot,t)\|^2_{L^2(\Omega)} = 0$$

"

$\hookrightarrow$ One has conservation of energy for (WV):

$$\frac{1}{2}\|u_t(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(\cdot,t)\|^2_{L^2(\Omega)} =$$

$$\underbrace{\frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_0'\|^2_{L^2(\Omega)}}_{\text{initial energy}} \quad \forall t.$$

2) Variational formulation and FE problem:

Consider the nonhomogeneous wave eq.

$$(W)\quad \begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ + \text{IC } u_0, v_0. \end{cases}$$

It should not be a surprise, that proceeding as 1d, one gets the VF for (W):

(VF) For each $0 < t \le T$, find $u(\cdot,t) \in H_0^1(\Omega)$ s.t.

$$\begin{cases} (u_{tt}(\cdot,t), v)_{L^2} + (\nabla u(\cdot,t), \nabla v(\cdot,t))_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in H_0^1 \\ u(x,0) = u_0(x) \end{cases}$$

$$u_t(x,0) = v_0(x)$$

As usual, one gets from (VF), the FE prob:

(FE) For each $0<t\leq T$, find $u_h(\cdot,t) \in V_h^0$ s.t.

$$\begin{cases} (u_{h,tt}(\cdot,t), v_h)_{L^2} + (\nabla u_h(\cdot,t), \nabla v_h)_{L^2} = (f(\cdot,t), v_h)_{L^2} \quad \forall v_h \in V_h^0 \\ u_h(x,0) = \pi_h u_0(x) \\ u_{h,t}(x,0) = \tilde{\pi}_h v_0(x) \end{cases}$$