(R)

- Ritz projection $R_h : H_0^1 \to V_h^0$

$$a(R_h v - v, v_h) = 0 \quad \forall v_h \in V_h^0$$

Here $a(u,v) = (\nabla u, \nabla v)_{L^2}$ energy i.p.

$V_h^0 \to$ FE space

$$\| R_h v - v \|_{L^2} \leq C \cdot h^2 \| v \|_{H^2}$$

- Error FEM for heat: $\| u_h(\cdot,t) - u(\cdot,t) \|_{L^2} \leq C \cdot h^2$

Idea: error = $err_1 + err_2$ $\to$ $err_1 \to$ direct using R_h; $\to err_2 \to$ find heat eq. for err_2 + stability

- (W) $\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(\cdot,0) = u_0,\ u_t(\cdot,0) = v_0 & \text{in } \Omega \end{cases}$

$f = 0 \to$ conservation of energy $\frac{1}{2} \| u_t(\cdot,t) \|_{L^2}^2 + \frac{1}{2} \| \nabla u(\cdot,t) \|_{L^2}^2$

(VF) $(u_{tt}(\cdot,t), v)_{L^2} + (\nabla u(\cdot,t), \nabla v)_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in H_0^1$
+ IC

(FE) $(u_{h,tt}(\cdot,t), v_h)_{L^2} + (\nabla u_h(\cdot,t), \nabla v_h)_{L^2} = (f(\cdot,t), v_h)_{L^2} \quad \forall v_h \in V_h^0$
+ IC

$V_h^0 = \operatorname{span}(\{\varphi_j\}_{j=1}^{n_i})$

- From the FE prob., as usual, we derive a syst. of ODE thanks to the following observations

$$u_h(x,t) = \sum_{j=1}^{n_i} \xi_j(t)\varphi_j(x) \quad \text{and take } v_h = \varphi_i .$$

Insert this in (FE) and get the syst.:

$$M\ddot{\xi}(t) + S\xi(t) = F(t)$$
$$\xi(0), \dot{\xi}(0),$$

$S \leftrightarrow$ stiffness matrix
$M \leftrightarrow$ mass matrix
$F \leftrightarrow$ vector with entry $(f(\cdot,t),\varphi_i)_{L^2}$

$\xi \leftrightarrow$ unknown.

- Last step is to use f.ex. C-N to get an approx $\xi^n \approx \xi(t_n)$, where $t_n = n\cdot k$, k time step.

$$\hookrightarrow \quad u_h^n(x,t_n) = \sum_{j=1}^{n_i} \xi_j^n \varphi_j(x) \approx u(x,t_n).$$

3) A priori error estimates for the semi-discrete problem:

Th: Let u, u_h be the exact sol. (VF) of wave equation, resp. to (FE). Assume u, u_h nice enough.
For $t \geq 0$ of interest, one has the error estimate

$$\|u(\cdot,t)-u_h(\cdot,t)\|_{L^2(\Omega)} \leq C\cdot\Big(|\Pi_h u_0 - R_h u_0|_1 + \|\underbrace{\tilde{\Pi}_h v_0 - R_h v_0}_{\Pi_h v_0 - v_0 + v_0 - R_h v_0}\|_{L^2}\Big)$$

$\pm v_0$

$$+\, C\cdot h^2 \underbrace{\left(\|u(\cdot,t)\|_{H^2} + \int_0^t \|u_{tt}(\cdot,s)\|_{H^2}\, ds\right)}_{\leq C \text{ since } u \text{ is nice enough}},$$

$$\leq C\cdot h^2$$

$\Pi_h u_0 \to$ cont. piw. linear interpolant of u_0
$R_h u_0 \to$ Ritz projection
$|v|_1 \to$ semi norm $\quad |v|_1 = \left(\sum_{|\alpha|=1} \|D^\alpha v\|_{L^2}^2\right)^{1/2}$

$\Big(f:\mathbb{R}\to\mathbb{R}: \ |f|_1 = \|f'\|_{L^2}$
$\ \ f:\mathbb{R}^2\to\mathbb{R}: \ |f|_1 = \|\nabla f\|_{L^2} \Big)$

Proof!

- We start by writing the error $u_h - u$ as

$$u_h - u = (u_h - R_h u) + (R_h u - u) =: \theta + \rho$$

- For the error term $\rho = R_h u - u$, we use the following

$$\|\rho(\cdot,t)\|_{L^2} + h\,|\rho(\cdot,t)|_1 \le C \cdot h^2 \|u(\cdot,t)\|_{H^2}$$

$$\|\rho_t(\cdot,t)\|_{L^2} \le C \cdot h^2 \, \|u_t(\cdot,t)\|_{H^2}$$

$$\|\rho_{tt}(\cdot,t)\|_{L^2} \le C \cdot h^2 \, \|u_{tt}(\cdot,t)\|_{H^2}$$ (see end proof)

- For the error term $\theta = u_h - R_h u$, we find an equation that is satisfied by θ (wave eq.) and use something related to the energy.

As for the heat eq., one shows that θ satisfies

$$(\theta_{tt}(\cdot,t), v_h)_{L^2} + (\nabla\theta(\cdot,t), \nabla v_h)_{L^2} = -(\rho_{tt}(\cdot,t), v_h)_{L^2} \quad \forall v_h \in V_h^0$$
$$\forall t \ge 0.$$

Choose now $v_h = \theta_t$ and get from the above:

$$\underbrace{(\theta_{tt}(\cdot,t), \theta_t(\cdot,t))_{L^2}}_{\frac{1}{2}\frac{d}{dt}(\theta_t(\cdot,t), \theta_t(\cdot,t))_{L^2}} + \underbrace{(\nabla\theta(\cdot,t), \nabla\theta_t(\cdot,t))_{L^2}}_{\frac{1}{2}\frac{d}{dt}(\nabla\theta(\cdot,t), \nabla\theta(\cdot,t))_{L^2}} = -(\rho_{tt}(\cdot,t), \theta_t(\cdot,t))$$

kinetic energy

potential energy

(see cons. energy)

That is, we get:

$$\frac{1}{2}\frac{d}{dt}\left(\|\theta_t(\cdot,t)\|_{L^2}^2 + \underbrace{\|\nabla\theta(\cdot,t)\|_{L^2}^2}_{|\theta(\cdot,t)|_1^2 \ \text{(semi-norm)}}\right) = -(\rho_{tt}(\cdot,t),\theta_t(\cdot,t))$$

Use C-S to get:

$$\frac{1}{2}\frac{d}{dt}\left(\ \cdots\ \right) \le \|\rho_{tt}(\cdot,t)\|_{L^2}\cdot\|\theta_t(\cdot,t)\|_{L^2}$$

Integrate in time $\int_0^t \ldots ds$:

$$\frac{1}{2}\left(\|\theta_t(\cdot,t)\|_{L^2}^2 + |\theta(\cdot,t)|_1^2 - \|\theta_t(\cdot,0)\|_{L^2}^2 - |\theta(\cdot,0)|_1^2\right) \le$$

$$\le \int_0^t \|\rho_{tt}(\cdot,s)\|_{L^2}\cdot\|\theta_t(\cdot,s)\|_{L^2}\,ds$$

Next, we get:

$$\|\theta_t(\cdot,t)\|_{L^2}^2 + |\theta(\cdot,t)|_1^2 \le \|\theta_t(\cdot,0)\|_{L^2}^2 + |\theta(\cdot,0)|_1^2 +$$

$$+ 2\int_0^t \|\rho_{tt}(\cdot,s)\|_{L^2}\underbrace{\|\theta_t(\cdot,s)\|_{L^2}}_{\le \max_{0\le s\le t}\|\theta_t(\cdot,s)\|_{L^2}}\,ds$$

On the equations

$$\|\theta_t(\cdot,t)\|_{L^2}^2 + \underbrace{|\theta(\cdot,t)|_1^2}_{\geq 0} \leq \|\theta_t(\cdot,0)\|_{L^2}^2 + |\theta(\cdot,0)|_1^2 +$$

$$+\, 2 \int_0^{\cancel{t}\; T} \|g_{tt}(\cdot,s)\|_{L^2}\, ds \cdot \max_{0\leq s\leq \cancel{t}\; T} \|\theta_t(\cdot,s)\|_{L^2}$$

$t \leq T$

$$\leq \|\theta_t(\cdot,0)\|_{L^2}^2 + |\theta(\cdot,0)|_1^2 + 2\left(\int_0^T \|g_{tt}(\cdot,s)\|_{L^2}\,ds\right)^2 +$$

$2xy \leq 2x^2 + \frac{1}{2}y^2$

$$+\, \frac{1}{2}\left(\max_{0\leq s\leq T} \|\theta_t(\cdot,s)\|_{L^2}\right)^2 \quad (*)$$

Observe that the above inequality is valid for all time of interest $t>0$, hence we get

$$\underbrace{\left(1-\frac{1}{2}\right)}_{\frac{1}{2}}\left(\max_{s\in[0,T]} \|\theta_t(s)\|_{L^2}\right)^2 \leq \|\theta_t(\cdot,0)\|_{L^2}^2 + |\theta(\cdot,0)|_1^2 +$$

$$+\, 2\left(\int_0^T \|g_{tt}(\cdot,s)\|_{L^2}\,ds\right)^2$$

We insert the above estimate into (*) and finally obtain:

$$\|\theta_t(\cdot,t)\|_{L^2}^2 + |\theta(\cdot,t)|_1^2 \leq 2\,\|\theta_t(\cdot,0)\|_{L^2}^2 +$$

$$+2|\theta(\cdot,0)|_1^2 + 4\left(\int_0^T \|\rho_{tt}(\cdot,s)\|_{L^2} ds\right)^2 \quad (**)$$

The above is valid for $t \in (0,T]$ and in particular for $t=T$, where T arbitrary.

- Last step: we put everything together:

$$\|u_h(\cdot,t) - u(\cdot,t)\|_{L^2} \overset{\Delta}{\leq} \|\theta(\cdot,t)\|_{L^2} + \|\rho(\cdot,t)\|_{L^2} \leq$$

Def of θ, ρ

Poincaré

$$\leq |\theta(\cdot,t)|_1 + \|\rho(\cdot,t)\|_{L^2} \overset{(**)}{\leq}$$

$|v|_1 = \|\nabla v\|_{L^2}$

$$\leq C \cdot \|\theta_t(\cdot,0)\|_{L^2} + C \cdot |\theta(\cdot,0)|_1 +$$

$$+ C \cdot \int_0^T \|\rho_{tt}(\cdot,s)\| ds + \|\rho(\cdot,t)\|_{L^2} \leq$$

$$\leq C \cdot \|u_{h,t}(\cdot,0) - R_h u_t(\cdot,0)\|_{L^2} + C \cdot |u_h(\cdot,0) - R_h u(\cdot,0)|_1 +$$

Def of $\theta = u_h - R_h u$, estimates for ρ, ρ_{tt} seen in beginning

$$+ \; C \cdot h^2 \int_0^T \| u_{tt}(\cdot, s) \|_{H^2} \, ds + C \cdot h^2 \| u(\cdot, t) \|_{H^2}$$

Def u_h and u

$$\leq C \cdot \| \Pi_h v_0 \overset{\pm v_0}{\vee} - R_h v_0 \|_{L^2} + C \cdot | \Pi_h u_0 \overset{\pm u_0}{\vee} - R_h u_0 |_1 +$$

$$+ \; C \cdot h^2 \int_0^T \| u_{tt}(\cdot, s) \|_{H^2} \, ds + C \cdot h^2 \| u(\cdot, t) \|_{H^2} .$$

... ouf

$\blacksquare$

Chapter XIII: The finite element

Goal: Study the concept of FE

1) Formal definition of a FE:

We begin by giving the def. of FE:

Def: (Ciarlet 1978) A finite element consists of the triplet (K, P, Σ), where

- $K \subset \mathbb{R}^d$ is a polygon

- P is a space of polynomials on K ($\dim(P) = n$)

- Σ is a unisolvent set of linear functionals on P: That is $\Sigma = \{L_1, L_2, \dots, L_n\}$ where $\dim(P) = n$ and $L_j : P \to \mathbb{R}$.

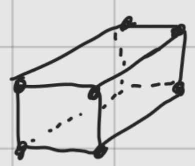
Rem: unisolvent means, more or less, that one can find n linearly independent polynomials $p_i \in P$ s.t. $L_j(p_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$

EX: • Each polygon comes from a mesh/triangulation of Ω. Mesh $T_h = \{K\}$.

K could be

1d  (interval)

2d  or  (quadrilaterals)

3d  (bricks)

etc

element domain

- P could be $P^{(1)}(K) \to$ polynomials of degree ≤ 1 defined on K

$P^{(2)}(K) \to \dots$ degree ≤ 2

Space of shape functions

- Σ_1: This gives a way to uniquely specify a basis / shape fct on each polygon K as well as the behaviour of these functions between adjacent polygons.
  → set of nodal values