(R) • Error estimate for FEM for wave eq.

• Def: Finite element (K, P, Σ)

$K \to$ element domain (think $\triangle$)

$P \to$ polyn. sp. / sp. of shape fct. (think $P^{(1)}(K)$)

$\Sigma = \{L_1, .., L_n\}$, $n = \dim(P)$, $L_j : P \to \mathbb{R}$ unisolvent

$\exists\ p_i \in P$ s.t. $\{p_i\}_{i=1}^{n}$ indep. and $L_j(p_i) = \delta_{ij}$

(think set of nodal values)

Ex: 1d Lagrange $P^{(e)}$ element:

Let $e > 0$ integer, $a, b \in \mathbb{R}$, $a < b$, and $e+1$ distinct points $a = x_0 < x_1 < x_2 < \dots < x_e = b$.

For K, we let $K = [a, b]$.

For P, we let $P = P^{(e)}(K) = \{$ set of polyn of degree $\leq e$ defined on $K\}$

For Σ, we let $\Sigma = \{L_0, L_1, .., L_e\}$ where $L_j : P \to \mathbb{R}$ are defined by $L_j(f) = f(x_j)$ $\forall f \in P$, $j = 0, 1, .., e$.

We now need to find polyn. λ_i, $i = 0, .., e$, s.t. $L_j(\lambda_i) = \delta_{ij}$ that is, by def of L_j,

$$\lambda_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases} \quad \text{for } i,j = 0,..,l.$$

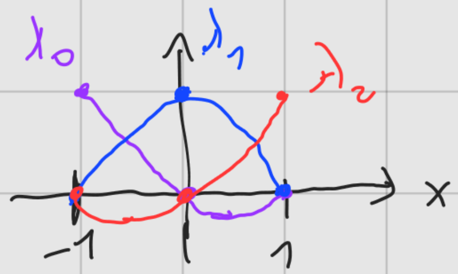
These are exactly Lagrange polynomials

$$\lambda_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{l} \frac{x - x_j}{x_i - x_j}$$

seen longtime ago.

$l = 1$

hat functions seen before



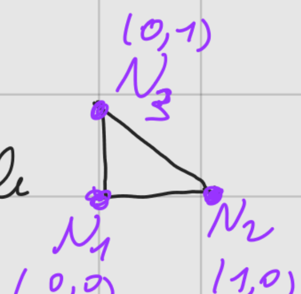
$l = 2$

see c(ii)(2) in Ass. 1.

↳ The polyn. $\lambda_i(x)$ are called <u>shape functions</u> and build a nodal basis of $P = P^{(l)}(K)$.

<u>Ex</u>: 2d linear Lagrange element:

$K \to$ take the reference triangle



$P \to P^{(1)}(K) = \{$ set of polyn. of degree ≤ 1 defined on $K\}$

$\Sigma \to \Sigma = \{L_1, L_2, L_3\}$ defined by $L_j(f) = f(N_j)$ for $f \in P$, $j = 1,2,3$ (N_j are nodes of K)

Let us now find shape functions using the conditions:

$$S_i(x,y) = a_i + b_i x + c_i y \quad \text{for } i = 1,2,3$$

$(S_i \in P)$

$$L_j(S_i) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

(unisolvent)

The above gives conditions in order to find a unique nodal basis which is given by

$$S_1^1(x,y) = 1 - x - y$$
$$S_2^1(x,y) = x$$
$$S^1(x,y) = y$$

$\Rightarrow$ These are the shape fct seen in chapter $\underline{\text{IX}}$

One can do much more than the above. For instance:

- 2d quadratic Lagrange element, one would take

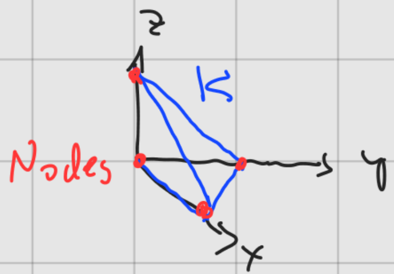
6 nodes and the shape fct are of form $S_i(x,y) = a_i + b_i x + c_i y + d_i xy + e_i x^2 + g_i y^2$

$S_1 = 1 - 3x - 3y + 2x^2 + 4xy + 2y^2$, $S_2 = 2x^2 - x$, $S_3 = 2y^2 - y$,
$S_4 = 4xy$, $S_5 = 4y - 4xy - 4y^2$, $S_6 = 4x - 4x^2 - 4xy$

- For 2d bilinear quadrilateral element, one takes

The shape fct will have the form
$S(x,y) = (a + bx)\cdot(c + dy)$

- For 3d linear Lagrange element, the element K is the tetrahedron;



Rem: • One can use other polynomials than Lagrange polyn.

Ex: Hermite polynomials

(enforce $\mathcal{C}^1$ continuity at the interface, instead of $\mathcal{C}^0$ continuity for Lagrange)

Ex: Morley element / Morley–Wang–Xu / MWX elements

K = ref. triangle

$L_i(f) = f(\text{node} \cdot)$ for nodes 1, 2, 3

$L_{i+3}(f) = n \cdot \nabla f(\text{node} \cdot)$ for nodes 4, 5, 6

Good for approximation in H^m in $\mathbb{R}^n$.

Used in solid mechanics (bridge)

2) Higher order FE:

Recall: For an integer $\ell \geq 1$, and a triangle $K \subset \mathbb{R}^2$, one recalls

$$\mathcal{P}^{(\ell)}(K) = \{ v : K \to \mathbb{R} : v(x,y) = \sum_{0 \leq i+j \leq \ell} c_{ij} x^i y^j \text{ for } (x,y) \in K, \ c_{ij} \in \mathbb{R} \}$$

Ex: $\ell = 2$: $\mathcal{P}^{(2)}(K) = \{ v : K \to \mathbb{R} : v(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 \}$

A polynomial $v \in \mathcal{P}^{(2)}(K)$ is determined by the values at nodes

Shape fct :

When considering a FEM for Poisson's eq., one works with the space

$$V_h^0 = \{ v \in C^0(\bar{\Omega}) : v_{|K} \in \mathbb{P}^{(\ell)}(K) \ \forall K \in \mathcal{T}_h \text{ and } v_{|\partial\Omega} = 0 \}$$

One then obtains error estimates of the form

<u>Th</u>: Let $\Omega \subset \mathbb{R}^2$ convex polygon. Let u be sol. Poisson's eq

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Let $u_h \in V_h^0$ the corresponding FE approximation. Assume that $u \in H^{\ell+1}(\Omega)$, then the error of FEM reads

$$\|u_h - u\|_{H^1(\Omega)} \leq C \cdot h^{\boxed{\ell}} \, |u|_{H^{\ell+1}(\Omega)}$$

(semi-norm)

3) Variational crimes :

Consider a variational problem

(VF) Find $u \in \mathcal{U}$ s.t. $a(u,v) = \ell(v) \ \forall v \in \mathcal{V}$.

In reality, one must work with the prob

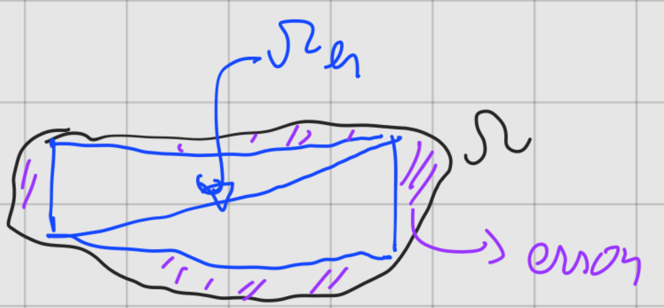
(FE) Find $u_h \in \mathcal{U}_h$ s.t. $a_h(u_h, v_h) = \ell_h(v_h) \ \forall v_h \in V_h$,

Where the index h denotes approximation errors, for instance:

→ From numerical integration, that is

$$\int_K f(x,y)\,dx\,dy \approx \frac{|K|}{3} f(0,0)$$ or

any other quadrature formula.

→ 

error in triangulation.

→ etc.

↳ Such errors are called variational crimes

→ Such errors are called variational crimes

(G. Strang 1972)

As a consequence, error estimates for FEM are much more involved!

Ex: $a(u,v) = \ell(v)$

$a(u,v) = \ell_h(v)$ ← error only in $\ell(\cdot)$

→ of order p

For a FEM for Poisson's eq., one gets

$$\|u-u_h\|_{H^1(\Omega)} \leq C \cdot h^p |u|_{H^{p+1}(\Omega)} + C \cdot \underbrace{\sup_{w \in V_h^0} \frac{|\ell(w) - \ell_h(w)|}{\|w\|_{H^1(\Omega)}}}$$

" $\approx h w + h^2 (w)^2 + \ldots$ "

Want this term to be of the size h^p

(linear element, $p=1$, h)

Chapter XIV: Finite difference approximation

Goal: Present finite difference (FD) for Poisson's eq. in 2d.

Recall:

Forward difference $y'(x) \approx \dfrac{y(x+h) - y(x)}{h}$

Backward difference: $y'(x) \approx \dfrac{y(x) - y(x-h)}{h}$

1) FD for Poisson's equation in 2d:

Want to compute numerical approximation of Poisson's eq:

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = [0,1] \times [0,1]$ unit square,

$f = f(x,y)$, $g = g(x,y)$ are given, $u = u(x,y) = ??$

Idea: Using FD to approximate Δu

Recall: $\Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y)$

One has:

Forward difference backward

$$u_{xx}(x,y) = \frac{\partial}{\partial x}\left(u_x(x,y)\right) \approx \frac{u_x(x+h,y)-u_x(x,y)}{h} \overset{\text{diff.}}{\approx}$$

$$\approx \frac{\frac{u(x+h,y)-u(x,y)}{h} - \frac{u(x,y)-u(x-h,y)}{h}}{h}$$