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Chapter 14: Finite difference approximation (summary)

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Goal: Present finite difference (FD) for Poisson's equation in $2d$.

- We recall forward and backward finite difference:
$$y'(x) \approx \frac{y(x+h)-y(x)}{h} \quad \text{and} \quad y'(x) \approx \frac{y(x)-y(x-h)}{h},$$
where $y\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function and $h>0$ is a given (small) real number.

- Let $\Omega = [0,1]\times[0,1] \subset \mathbb{R}^2$ be the unit square, $f, g\colon \Omega \to \mathbb{R}$ nice functions. Consider Poisson's equation
$$\begin{cases} \Delta u = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$
where $u = u(x,y)$ is the unknown function and $\Delta u = u_{xx} + u_{yy}$ is the Laplacian.

  To find a numerical approximation to the solution u of Poisson's equation, consider a mesh size $h = \frac{1}{n+1}$ for some (large) integer n and the grid $x_i = ih$ and $y_j = jh$ for $i, j = 1, \ldots, n$.

  We first approximate the derivatives in the Laplacian using forward and backward finite difference:
$$u_{xx} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$$
resulting in the discrete Laplace operator Δ_h
$$\Delta u(x,y) \approx \Delta_h u(x,y) = \frac{u(x+h,y) + u(x,y+h) - 4u(x,y) + u(x-h,y) + u(x,y-h)}{h^2}.$$
Denote then $u_{ij} \approx u(x,y)$ at the grid point (x_i, y_j) and $f_{ij} = f(x_i, y_j)$, resp. $g_{ij} = g(x_i, y_j)$, one gets the linear system of equations
$$\Delta_h u_{ij} = f_{ij} \quad \text{for} \quad i, j = 1, \ldots, n.$$
Using the boundary condition, one then ends up with the linear system of equations
$$A\mathbf{u} = F,$$
with a block tridiagonal matrix A of size $n^2 \times n^2$, a vector F of size $n^2 \times 1$ (containing f_{ij} and g_{ij}) and the unknown vector
$$\mathbf{u} = (u_{11}, u_{12} \ldots, u_{1n}, \ldots, u_{n1}, \ldots, u_{nn})^T.$$

- Under some assumptions, one can prove convergence of the above finite difference approximation to the solution to Poisson's equation on the unit square:
$$\left\| u(x_i, y_j) - u_{ij} \right\|_\infty \le Ch^2.$$

- The above can be generalised in various directions, for instance: high order FD, Neumann or Robin BC, general operator instead of the Laplacian, etc.

Further resources:

- $1d$ Poisson FD solver at tum.de
- FD for $1d$ Poisson and heat eq. at unibs.it
- FD for Poisson in $2d$ at youtube.com
- FD for Poisson in $1d$ and 2d at sc.fsu.edu.edu