46) Consider the non-homogeneous heat equation in 1D with variable coefficient

$$\begin{cases} \dot{u}(x,t) - (a(x,t)\,u'(x,t))' = f(x,t), & x \in (0,1),\ t>0 \\ u(0,t) = u(1,t) = 0, & t>0 \\ u(x,0) = u_0(x) \end{cases}$$

Formulate a cG(1)dG(0) finite element method for this problem.

Solution

Multiply by $v = v(x,t)$, integrate over $(0,1)$, integrate by parts, integrate over an interval $I_n = (t_{n-1}, t_n]$.

Then we get the VF:

$$\int_{I_n}\int_0^1 (\dot{u}v + au'v')\,dx\,dt = \int_{I_n}\int_0^1 fv\,dx\,dt, \quad \forall v(x,t) : v(0,t) = v(1,t) = 0$$

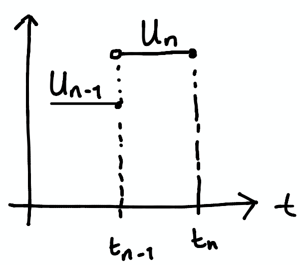
The cG(1)dG(0) method:

For each time interval I_n of length k_n, let $U(x,t) = U_n(x)$

$$U_n(x) = \sum_{j=1}^{M} \xi_{n,j}\varphi_j(x)$$

$$U_n'(x) = \sum_{j=1}^{M} \xi_{n,j}\varphi_j'(x)$$

$$\dot{U} = 0 + (U_n - U_{n-1})\delta_{t_{n-1}}$$



Test function in t : $\phi_n = 1$

Test function in x : φ_i usual hat functions

$$VF \Rightarrow \int_{I_n}\int_0^1 aU_n'\varphi_i'\,dx\,dt + \int_{I_n}\int_0^1 (U_n - U_{n-1})\underline{\delta_{t_{n-1}}}\varphi_i\,dx\,dt = \int_{I_n}\int_0^1 f\varphi_i\,dx\,dt$$

Def: $\int_{I_n}\delta_{t_{n-1}} = 1$

$$\Rightarrow \int_{I_n}\int_0^1 aU_n'\varphi_i'\,dx\,dt + \int_0^1 (U_n - U_{n-1})\varphi_i\,dx = \int_{I_n}\int_0^1 f\varphi_i\,dx\,dt$$

$$\Rightarrow \sum_{j=1}^{M} \xi_{n,j} \underbrace{\int_{I_n}\int_0^1 a(x,t)\varphi_j'\varphi_i'\,dx\,dt}_{=A\text{ (dep. on } t!)} + \sum_{j=1}^{M}(\xi_{n,j}-\xi_{n-1,j})\underbrace{\int_0^1 \varphi_j\varphi_i\,dx}_{=M} = \underbrace{\int_{I_n}\int_0^1 f\varphi_i\,dx\,dt}_{b}$$

$$\underline{(A(t)+M)\xi_n = M\xi_{n-1}+b}$$

10.10) Consider the convection problem

$$\begin{cases} \dot{u}+\beta\cdot\nabla u + \alpha u = f, & x\in\Omega,\ t>0 \\ u=g, & x\in\Gamma_-,\ t>0 \\ u(x,0)=u_0(x), & x\in\Omega \end{cases}$$

with $\partial\Omega = \Gamma_+\cup\Gamma_-$. Assume that $\alpha-\frac{1}{2}\nabla\cdot\beta \geq c>0$. Show the following stability estimate:

$$\|u(\cdot,T)\|^2 + c\int_0^T \|u(\cdot,t)\|^2dt + \int_0^T\int_{\Gamma_+} n\cdot\beta\, u^2\,ds\,dt \leq$$

$$\leq \|u_0\|^2 + \frac{1}{c}\int_0^T\|f(\cdot,t)\|^2dt + \int_0^T\int_{\Gamma_-}|n\cdot\beta|\,g^2\,ds\,dt,$$

where $\|u(\cdot,T)\|^2 = \int_\Omega u(x,T)^2\,dx$.

Solution

Multiply DE by u and integrate over Ω:

$$(*)\quad \underbrace{\int_\Omega u\dot{u}}_{(**)} + \underbrace{\int_\Omega(\beta\cdot\nabla u)u}_{=\frac{1}{2}\int_\Omega \beta\cdot\nabla(u^2)} + \int_\Omega \alpha u^2 = \int_\Omega fu$$

Green's:
$$\int_\Omega \underset{\nabla\cdot\beta}{\underline{\Delta u}}\ \underset{u^2}{\underline{v}} = \int_{\partial\Omega}\underset{\beta}{\underline{(\nabla u\cdot n)}}\ \underset{u^2}{\underline{v}} - \int_\Omega \underset{\beta}{\underline{\nabla u}}\cdot\underset{\nabla(u^2)}{\underline{\nabla v}}$$

$$\frac{1}{2}\int_\Omega \beta\cdot\nabla(u^2) = \frac{1}{2}\int_{\partial\Omega}(\beta\cdot n)u^2 - \frac{1}{2}\int_\Omega u^2\nabla\cdot\beta$$

Also using $\dot{u}u = \frac{1}{2}\frac{d}{dt}(u^2)$, we get $(**) = \frac{1}{2}\frac{d}{dt}\int_\Omega u^2$

$$(*) \Rightarrow \frac{d}{dt}\int_\Omega u^2 + \int_{\Gamma_+} u^2(\beta\cdot n) + \int_{\Gamma_-} g^2(\beta\cdot n) \underline{- \int_\Omega u^2 \nabla\cdot\beta + 2\int_\Omega \alpha u^2} = 2\int_\Omega f u$$

$$\frac{d}{dt}\|u\|^2_{L^2} + \int_\Omega \underbrace{(2\alpha - \nabla\cdot\beta)}_{\geq 2c > 0} u^2 + \int_{\Gamma_+} u^2(\beta\cdot n) \leq -\int_{\Gamma_-} g^2(\beta\cdot n) + 2\int_\Omega f u$$

$$\frac{d}{dt}\|u\|^2_{L^2} + 2c\|u\|^2_{L^2} + \int_{\Gamma_+} u^2(\beta\cdot n)\,ds \leq \int_{\Gamma_-} g^2|\beta\cdot n| + 2\int_\Omega f u$$

$$\left\{ 2fu \leq \left(\frac{f}{\sqrt{c}}\right)^2 + (\sqrt{c}\,u)^2 \right\}$$

$$\Rightarrow \frac{d}{dt}\|u\|^2_{L^2} + c\|u\|^2_{L^2} + \int_{\Gamma_+} u^2(\beta\cdot n)\,ds \leq \frac{1}{c}\|f\|^2_{L^2} + \int_{\Gamma_-} g^2|\beta\cdot n|\,ds$$

Integrate this w.r.t t from 0 to T:

$$\underbrace{\int_0^T \frac{d}{dt}\|u\|^2_{L^2}}_{=\left[\|u\|^2_{L^2}\right]_0^T} + c\int_0^T \|u\|^2_{L^2}\,dt + \int_0^T\int_{\Gamma_+} u^2(\beta\cdot n)\,ds\,dt \leq$$

$$\leq \frac{1}{c}\int_0^T \|f(\cdot,t)\|^2_{L^2}\,dt + \int_0^T\int_{\Gamma_-} g^2|\beta\cdot n|\,ds\,dt$$

$$\|u(\cdot,T)\|^2_{L^2} + c\int_0^T \|u\|^2_{L^2}\,dt + \int_0^T\int_{\Gamma_+} u^2(\beta\cdot n)\,ds\,dt \leq$$

$$\leq \|u_0\|^2_{L^2} + \frac{1}{c}\int_0^T \|f(\cdot,t)\|^2_{L^2}\,dt + \int_0^T\int_{\Gamma_-} g^2|\beta\cdot n|\,ds\,dt$$

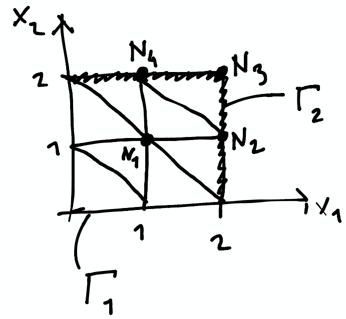
$\checkmark$

From the exam 2017-03-15

4) In the square domain $\Omega = (0,2)^2$ with boundary $\Gamma = \partial\Omega$, consider

$$\begin{cases} -\Delta u + u = 1, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_1 := \Gamma \setminus \Gamma_2 \\ \left.\frac{\partial u}{\partial x_1}\right|_{x_1=2} = \left.\frac{\partial u}{\partial x_2}\right|_{x_2=2} = 1, & \text{on } \Gamma_2 := \{x_1 = 2\} \cup \{x_2 = 2\} \end{cases}$$

Determine stiffness matrix and mass matrix and load vector of the cG(1) FEM with the triangulation:



$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Solution $\quad h = 1.$

Test function space $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}$

Multiply the DE by $v \in V$ and integrate:

$$(*) \quad -(\Delta u, v) + (u, v) = (1, v) \qquad \forall v \in V$$

Green's: $-(\Delta u, v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds =$

$$= \{v = 0 \text{ on } \Gamma_1\} = (\nabla u, \nabla v) - (n \cdot \nabla u, v)_{\Gamma_2} =$$

$$= \left\{ \left.\frac{\partial u}{\partial x_1}\right|_{x_1=2} = \left.\frac{\partial u}{\partial x_2}\right|_{x_2=2} = 1 \right\} = (\nabla u, \nabla v) - (1, v)_{\Gamma_2}$$

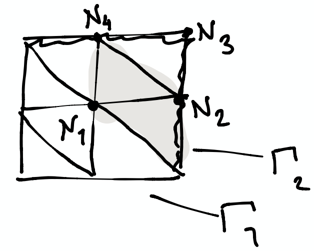
$$(*) \Rightarrow (\nabla u, \nabla v) + (u, v) = (1, v) + (1, v)_{\Gamma_2} \qquad \forall v \in V$$

Let $u_h(x) = \sum_{j=1}^{4} \xi_j \varphi_j(x)$, φ_j standard tent functions.

$$\Rightarrow \sum_{j=1}^{4} \xi_j \Big(\underbrace{\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx}_{S} + \underbrace{\int_\Omega \varphi_i \varphi_j \, dx}_{M} \Big) = \underbrace{\int_\Omega \varphi_i \, dx + \int_{\Gamma_2} \varphi_i \, ds}_{b}, \quad i = 1, \dots, 4$$

or $(S+M)\xi = b$

Assemble global matrices M and S:



$M_{11} = 2m_{11} + 4m_{22} = \frac{6}{12}h^2$

$M_{12} = M_{14} = 2m_{12} = \frac{1}{12}h^2$

$M_{13} = 0$

$M_{22} = M_{44} = m_{11} + 2m_{22} = \frac{3}{12}h^2$

$M_{23} = M_{34} = m_{12} = \frac{1}{24}h^2$

$M_{24} = 2m_{23} = \frac{1}{12}h^2$

$M_{33} = m_{11} = \frac{1}{12}h^2$

$S_{11} = 2s_{11} + 4s_{22} = 4$

$S_{12} = \dots = -1$

$S_{13} = 0$

$S_{22} = S_{44} = \dots = 2$

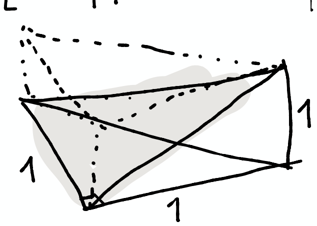
$S_{23} = S_{34} = -\frac{1}{2}$

$S_{24} = \dots = 0$

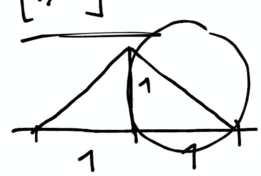
$S_{33} = \dots = 1$

$$M = \frac{h^2}{24} \begin{bmatrix} 12 & 2 & 0 & 2 \\ 2 & 6 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 2 & 2 & 1 & 6 \end{bmatrix} \qquad S = \begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 2 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & 0 & -\frac{1}{2} & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} (1,\varphi_1) + (1,\varphi_1)_{\Gamma_2} \\ \vdots \\ (1,\varphi_4) + (1,\varphi_4)_{\Gamma_2} \end{bmatrix} = \begin{bmatrix} 6\cdot\frac{1}{6} + 0 \\ 3\cdot\frac{1}{6} + 2\cdot\frac{1}{2} \\ 1\cdot\frac{1}{6} + 2\cdot\frac{1}{2} \\ 3\cdot\frac{1}{6} + 2\cdot\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 7/6 \\ 3/2 \end{bmatrix}$$



$\text{Vol} = \frac{1\cdot 1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}$



Area: $\frac{1\cdot 1}{2} = \frac{1}{2}$

10.4) Formulate the equation for cG(1)dG(1) for the two-dimensional heat equation using the discrete Laplacian.

Solution

2-dimensional heat equation:

$$\begin{cases} \dot{u}(x,t) - \Delta u(x,t) = f(x,t), & x \in \Omega,\ t > 0 \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

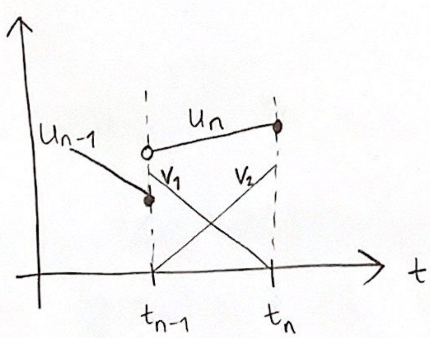
Let $I_n = (t_{n-1}, t_n]$. FEM formulation: Find u such that

$$\int_{I_n} \big((\dot{u}, w) + (\nabla u, \nabla w) \big)\, dt = \int_{I_n} (f, w)\, dt \quad \forall\, w \in W_{k_n}^{(1)}$$

$W_{k_n}^{(1)} = \{ w :$ w p.w. linear, continuous in space, p.w. linear, discontinuous in time $\}$

Consider one space-time slab S_n.

Let $$u_n(x,t) = \phi_n(x) \underbrace{\frac{t_n - t}{k_n}}_{=: v_1(t)} + \psi_n(x) \underbrace{\frac{t - t_{n-1}}{k_n}}_{=: v_2(t)}$$



We'll use the discrete laplacian and the projection formulation: $(\nabla u, \nabla w) = -(\Delta_h u, w)$, $(P_h f, w) = (f, w)$ $\forall w$ (see problem 38)

$\Rightarrow$ FEM formulation: $\int_{I_n} (\dot{U} v_i - \Delta_h U v_i)\, dt = \int_{I_n} P_h f v_i\, dt$, $i = 1,2$

using v_1 and v_2 as test functions in t.

We have that

$$\dot{U}_n(x,t) = -\frac{1}{k_n}\phi_n(x) + \frac{1}{k_n}\psi_n(x) + (\phi_n(x) - P_h\psi_{n-1})\,\delta_{t_{n-1}}$$

Plug into FEM formulation $\Rightarrow$

<u>$i=1$:</u>

$$\int_{I_n}\Big(\underbrace{\frac{1}{k_n}(\psi_n-\phi_n)}_{\text{Indep. of } t} v_1 + (\phi_n - P_h\psi_{n-1})\underbrace{\delta_{t_{n-1}} v_1}_{\text{By def,}} - \underbrace{\Delta_h(\phi_n v_1 + \psi_n v_2)}\, v_1\Big)dt = \int_{I_n} P_h f v_1\, dt$$

$\int_{I_n} v_1\, dt = \frac{k_n}{2} = \int_{I_n} v_2\, dt$

$\int_{I_n} \delta_{t_{n-1}} v_1 = v_1(t_{n-1}) = 1$

$\int_{I_n} v_1 v_1 = \frac{k_n}{3} = \int_{I_n} v_2 v_2$

$\int_{I_n} v_1 v_2 = \frac{k_n}{6}$

$$\Rightarrow \frac{1}{k_n}(\psi_n - \phi_n)\frac{k_n}{2} + \phi_n - P_h\psi_{n-1} - \frac{k_n}{3}\Delta_h\phi_n - \frac{k_n}{6}\Delta_h\psi_n = \int_{I_n} P_h f\,\frac{t_n - t}{k_n}\,dt$$

$$\Rightarrow \frac{1}{2}(\psi_n - \phi_n) + \phi_n - P_h\psi_{n-1} - \frac{k_n}{6}(2\Delta_h\phi_n + \Delta_h\psi_n) = \int_{I_n} P_h f\,\frac{t_n - t}{k_n}\,dt$$

<u>$i=2$:</u> $$\int_{I_n}\Big(\frac{1}{k_n}(\psi_n-\phi_n)v_2 + (\phi_n - P_h\psi_{n-1})\underbrace{\delta_{t_{n-1}} v_2}_{\int_{I_n}\delta_{t_{n-1}} v_2 = v_2(t_{n-1}) = 0} - \Delta_h(\phi_n v_1 + \psi_n v_2)\, v_2\Big)dt = \int_{I_n} P_h f v_2\, dt$$

$$\Rightarrow \frac{1}{2}(\psi_n - \phi_n) - \frac{k_n}{6}(\Delta_h\phi_n - 2\Delta_h\psi_n) = \int_{I_n} P_h f\,\frac{t - t_{n-1}}{k_n}\,dt$$

So our cG(1)dG(1)-formulation becomes:

$$\begin{cases} \left(\frac{1}{2} - \frac{k_n}{6}\Delta_h\right)\psi_n + \left(\frac{1}{2} - \frac{k_n}{3}\Delta_h\right)\phi_n = P_h\psi_{n-1} + \int_{I_n} P_h f\,\frac{t_n - t}{k_n}\,dt \\ \left(\frac{1}{2} - \frac{k_n}{3}\Delta_h\right)\psi_n - \left(\frac{1}{2} - \frac{k_n}{6}\Delta_h\right)\phi_n = \int_{I_n} P_h f\,\frac{t - t_{n-1}}{k_n}\,dt \end{cases}$$

10.16) Consider the IBVP

$$\begin{cases} \ddot{u} - \Delta u + u = 0, & x \in \Omega, t > 0 \\ u = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), & x \in \Omega \\ u(x,1) = u_1(x), & x \in \Omega \end{cases}$$

Rewrite the problem as a system of two equations with a time derivative of order at most 1. (Why this modification?)

Solution

Let $v = \dot{u}$. Then $\dot{v} = \ddot{u} = \Delta u - u$

Let $w = (u, v)^T$.

Then $\dot{w} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ \Delta u - u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{bmatrix} w$

Easier to compute such a problem.

%.