R

- FE (K, P, Σ)

$K \to \Delta$, $\quad P \to \mathbb{P}^{(1)}(K)$, $\quad \Sigma \to L_i(\psi) = f(x_i)$ f. ex (shape fct.)

- FD for Poisson on $\Omega = [0,1] \times [0,1]$:

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

f, g given, $\Delta u = u_{xx} + u_{yy}$, $u = u(x,y) = ??$

FD approximation of u?

- Quiz, feedback

forward Diff.   backward Diff

$$u_{xx}(x,y) = \frac{\partial}{\partial x}\left(u_x(x,y)\right) \approx \frac{u_x(x+h,y) - u_x(x,y)}{h} \approx$$

$$\approx \frac{\frac{u(x+h,y) - u(x+h-h,y)}{h} - \frac{u(x,y) - u(x-h,y)}{h}}{h} \approx$$

$$\approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$$

Similiarly, we also get:

$$u_{yy} \approx \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

The above hence provides an approximation of Δu called the discrete Laplace operator (5-point star)

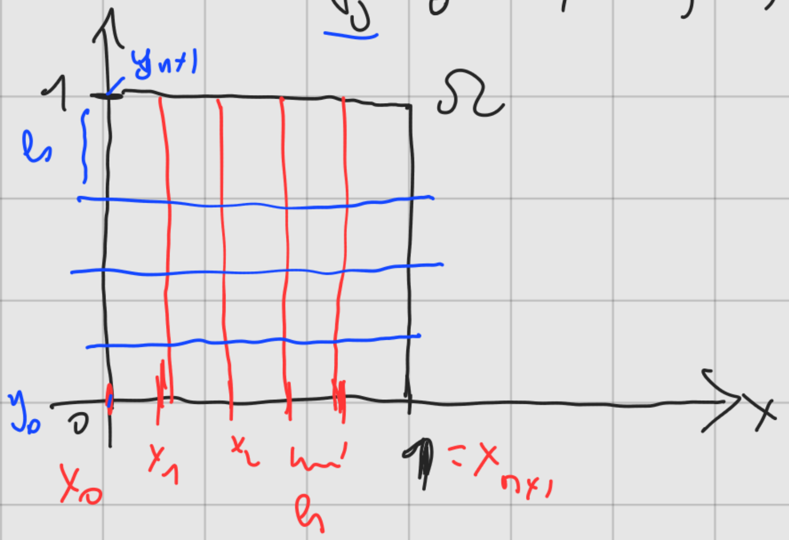
$$\Delta_h u(x,y) = \frac{u(x+h,y) + u(x,y+h) - 4u(x,y) + u(x-h,y) + u(x,y-h)}{h^2}$$

In order to find a FD approx. of the solution to Poisson, we set a mesh size $h := \frac{1}{n+1}$, for $n \in \mathbb{N}$ (large) and grids;

$$x_i := i \cdot h \quad \text{for } i = 0,1,2,\dots,n+1$$

$$y_j := j \cdot h \quad \text{for } j = 0,1,2,\dots,n+1$$

$y_j = jh$ for $j = 0, 1, \dots, n+1$



Define $f_{ij} := f(x_i, y_j)$ and $g_{ij} = g(x_i, y_j)$.

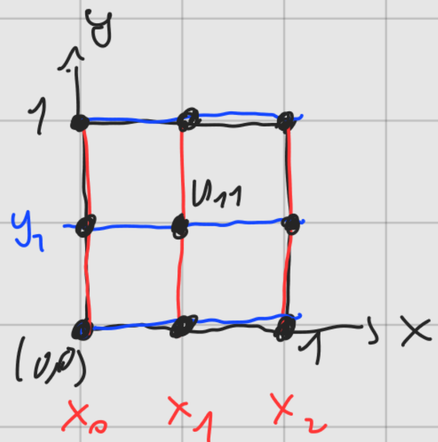
Set $u_{ij} \approx u(x_i, y_j)$ (approx. to exact sol. at grid point (x_i, y_j)):

$$\Delta_h u_{ij} = f_{ij}$$

Obs: This is a linear system of equations.

Ex: $n = 1 \rightsquigarrow h = \frac{1}{2}$, $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$

$y_0 = 0, y_1 = \frac{1}{2}, y_2 = 1$

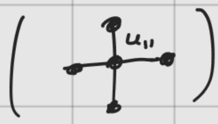


$\Delta_h u_{11} = f_{11}$ corresponds

(using def of Δ_h):

$\underbrace{}_{\text{Known }(*)}$ $\overbrace{}^{\text{unknown!!}}$ $\overbrace{}^{\text{known }(*)}$ known

$$\frac{u_{21}+u_{12}-4u_{11}+u_{01}+u_{10}}{h^2} = f_{11}$$



h^2 known

(*) $\to$ due to the BC $u = g$ on $\partial\Omega$

The above is a "system of linear eq." for u_{11} and provides the FD approx!

$$u_{11} = \left(h^2 f_{11} - (g_{21}+g_{12}+g_{01}+g_{10})\right)/(-4)$$

For $n>1$, one gets the following linear syst. of eq.:

$$AU = F,$$

where

$$A = \frac{1}{h^2}\left(\begin{array}{c|c|c} \begin{matrix} -4 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -4 \end{matrix} & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \begin{matrix} -4 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -4 \end{matrix} & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ & \ddots & \ddots \end{array}\right)$$

$$\left(\quad \mathcal{O} \quad \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \quad \begin{bmatrix} -4 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -4 \end{bmatrix} \right)$$

block tridiagonal matrix $(n^2 \times n^2)$

$$U = \begin{pmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1n} \\ u_{21} \\ \vdots \\ u_{2n} \\ u_{n1} \\ \vdots \\ u_{nn} \end{pmatrix} \quad \text{unknown vector } (n^2 \times 1)$$

$$F = \begin{pmatrix} f_{11} - \frac{1}{h^2}(p_1 + q_1) \\ f_{12} - \frac{1}{h^2} q_2 \\ \vdots \\ f_{1n} - \frac{1}{h^2}(q_n + r_1) \\ f_{21} - 1/h^2 \, p_2 \\ f_{22} \\ \vdots \\ f_{2n} - 1/h^2 \, r_2 \\ \vdots \\ f_{nn} - \frac{1}{h^2}(r_n + s_n) \end{pmatrix}$$

where

$p_j = p(x_j)$, $q_j = q(x_j)$,

$r_j = r(x_j)$, $s_j = s(x_j)$

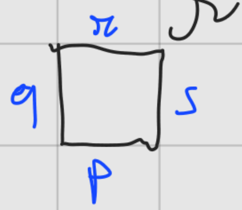
and

$u(x,0) = p(x)$

$u(0,y) = q(y)$ these are values at the

$u(x,1) = \pi(x)$ boundary $\partial\Omega$

$u(1,y) = s(y)$



Solving the linear syst. $A\mathcal{U} = F$ provides a FD approx. of u, the sol. to Poisson's eq. on square.

2) Convergence of FD:

Th. Assume that the sol. u to Poisson'eq. on unit square is in $C^4(\Omega)$. Denote the above FD approximation
↳ cont. diff.
by $u_{ij} \approx u(x_i, y_j)$. Then, one has the error estimate

$$\| u(x_i, y_j) - u_{ij} \|_\infty \leq C \cdot \boxed{h^2},$$

→ 0 as $h \to 0$!

where $\| w \|_\infty = \max\limits_{1 \leq i,j \leq n} |w_{ij}|$

Main ingredients for proof:

exact sol — numerical approx

1) Truncation error: $\| \Delta_h u - \Delta_h \mathcal{U} \|_\infty \leq \ldots \leq C \cdot h^2$

discrete Laplace op.

2) Discrete maximum principle

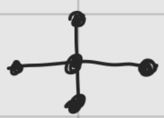
(If $\Delta_h u = 0 \Rightarrow u$ has it's max/min on $\partial\Omega, \dots$)

3) Poincaré-type inequality: $\|w\|_\infty \leq \frac{1}{8} \|\Delta_h w\|_\infty$

$$\|u(x_i, y_j) - u_{ij}\|_\infty \overset{(3)}{\leq} C \cdot \|\Delta_h (u - u_{ij})\|_\infty \overset{(1)}{\leq} C \cdot h^2$$

3) Generalisations:

- Higher order FD: need more information!



order 2

order 4

- Neumann or Robin BC OK.

(1D) $u'(0) = g$ $\quad u'(0) + u(0) = g$

- Laplacian $\Delta \rightsquigarrow$ more general operator

(1D) $L(u) = u_{xx}(x) - u_x(x) + f(u, x)$

- Possible extension for other domains than

, rectangle

FD	"vs"	FEM
classical/strong solutions		variational/weak solutions
approximate of equation		approximate solution
"easy" to implement		"difficult" to implement
"difficult" for complicated prob.		"easy" for complicated prob