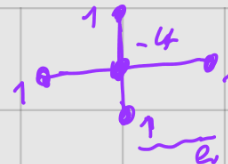
(R) FD for Poisson in unit square:



→ linear system "$Ax=b$" ⇝ error estimate

$\| u(x_i, y_j) - u_{ij} \|_\infty \leq C \cdot h^2$

• FD "vs" FEM.

Old exam from 15.03.21

2. Consider the following pseudo code

```
1 input: t0, y0, T, n
2   t = zeros(n+1,1); y = zeros(n+1,1);
3   t(1) = t0; y(1) = y0; h = (T-t0)/n;
4   for i = 1:n
5     t(i+1) = t(i)+h;
6     y(i+1) = y(i)+h*(y(i))^2;
7   end
8 output: t,y
```

Suppose that the input values are $t0 = 0, y0 = 1, T = 1$, and $n = 10$.

(a) What is the initial-value problem being approximated numerically? (1p)

(b) What is the numerical method being used? (1p)

(c) What is the numerical value of the step size? (1p)

(d) What are the first values of the output t, y? (1p)

a) Code at line 5+6: $y(i+1) = y(i) + h \cdot (y(i))^2$ ⇝

Explicit Euler for an IVP: $\begin{cases} y'(t) = y(t)^2 & 0 < t \leq T = 1 \\ y(0) = 1 & \text{(line 3)} \end{cases}$

b) Explicit Euler, see above.

c) From line 3, we get $h = \frac{T - t_0}{n} = \frac{1-0}{10} = \frac{1}{10}$

d) $t(1) = 0$, $t(2) \overset{\text{line 5}}{=} t(1) + h = 0 + \frac{1}{10} = \frac{1}{10}, \ldots$

$y(1) = 1$, $y(2) = y(1) + h\, y(1)^2 = 1 + \frac{1}{10} \cdot 1^2 = \frac{11}{10}, \ldots$

line 3

Old exam from

09.06.21

5. Let V be an Hilbert space with inner product and norm denoted by $(\cdot,\cdot)_V$ and $\|\cdot\|_V$. On this space, let a bilinear form $a(\cdot,\cdot)$ and a functional $\ell(\cdot)$ verifying the assumptions of Lax–Milgram that we recall: There exist $\alpha > 0, \beta \geq 0, \kappa > 0$ such that

$$|a(u,v)| \leq \alpha \, \|u\|_V \, \|v\|_V \qquad \forall u, v \in V$$

$$a(u,u) \geq \kappa \, \|u\|_V^2 \qquad \forall u \in V$$

$$|\ell(v)| \le \beta \|v\|_V \quad \forall v \in V.$$

Consider the variational problem: Find $u \in V$ such that

$$a(u,\varphi) = \ell(\varphi) \quad \forall \varphi \in V.$$

Let now $V_h \subset V$ be a finite dimensional subspace of V and $u_h \in V_h$ the solution to Galerkin's equation

$$a(u_h,\varphi_h) = \ell(\varphi_h) \quad \forall \varphi_h \in V_h.$$

(a) Show that the discrete solution u_h exists and is unique in V_h. (2p)

(b) Show the following bound

$$\|u - u_h\|_V \le \frac{\alpha}{\kappa} \|u - \varphi_h\|_V$$

a) Since the FE space $V_h \subset V$ Hilbert, then we can also use Lax-Milgram to show $\exists!$ sol. u_h of the FE prob.

Find $u_h \in V_h$ s.t. $a(u_h, \varphi_h) = \ell(\varphi_h)$ $\forall \varphi_h \in V_h$.

[since "a" and "ℓ" verify Lax-Milgram]

[Applyied L.M to show $\exists!$ sol. to Poisson' eq.

$$\begin{cases} -u''(x) = f(x) \\ u(0) = u(1) = 0. \end{cases}$$

For this probl. the space of interest $H_0^1(0,1)$.]

b) First:

VF tells us $a(u,\varphi) = \ell(\varphi)$ $\forall \varphi \in V$

Galerkin' eq. tells us $a(u_h, \varphi_h) = \ell(\varphi_h)$ $\forall \varphi_h \in V_h$ $(*)$

But $V_h \subset V$, in particular, one can take $\varphi = \varphi_h$ in VF, That is $\quad a(u, \varphi_h) = \ell(\varphi_h) \quad (**)$

Substracting $(*)$ from $(**)$, we get

$$a(u - u_h, \varphi_h) = 0 \quad \forall \varphi_h \in V_h.$$

Next:

$$\gamma_i \cdot \|u - u_h\|_V^2 \underset{\text{assumpt. in LM}}{\leq} a(u - u_h, u - u_h) \overset{\text{linearity}}{=} a(u - u_h, u) -$$

$$- \underbrace{a(u - u_h, u_h)}_{= 0 \text{ by first part}} =$$

$$= a(u - u_h, u) - \underbrace{a(u - u_h, \varphi_h)}_{= 0 \text{ by first part for } \varphi_h \in V_h}$$

$$= a(u - u_h, u - \varphi_h) \underset{\text{assumpt. LM}}{\leq} \alpha \|u - u_h\|_V \cdot \|u - \varphi_h\|_V$$

$$(\Rightarrow) \quad \|u - u_h\|_V \leq \frac{\alpha}{\gamma_i} \|u - \varphi_h\|_V \quad \forall \varphi_h \in V_h$$

(related to best approx.)

Old exam from 26.08.21

7. Let $A < B$ and denote $I =]A, B[$. Consider the variational problem: Find $u \in H^1(I)$ such that
$$a(u, v) = (f, v) \quad \text{for all } v \in H^1(I)$$

and the corresponding cG(1) approximation: Find $u_h \in V_h \subset H^1(I)$ such that
$$a(u_h, v) = (f, v) \quad \text{for all } v \in V_h.$$

One assumes that the bilinear form a is continuous, coercive and bounded on $H^1(I)$ and that the given function f is nice enough.

Denote by $\Pi_h : H^1(I) \to V_h$ the interpolation operator.

(a) Show that $a(u_h - u, v) = 0$ for all $v \in V_h$. (1p)

(b) Next, show that $\|\Pi_h u - u_h\|_{H^1(I)} \le C\|\Pi_h u - u\|_{H^1(I)}$. (2p)

(c) Finally, use the triangle inequality to get the estimate $\|u - u_h\|_{H^1(I)} \le C\|\Pi_h u - u\|_{H^1}$. (1p)

a) For $v_h \in V_h \subset H^1$, we have: $a(u, v_h) \overset{VF}{=} (f, v_h) \overset{FE/cG(1)}{=} a(u_h, v_h)$

This gives us $a(u_h - u, v_h) = 0 \quad \forall v_h \in V_h$.

b) $\alpha \|\Pi_h u - u_h\|^2_{H^1} \le a(\Pi_h u - u_h, \Pi_h u - u_h) + \underbrace{a(u_h - u, \overbrace{\Pi_h u - u_h}^{\in V_h})}_{= 0 \text{ by a)}}$

Coercivity of bilinear form $a(\cdot,\cdot)$

$= a(\Pi_h u - u, \Pi_h u - u_h) \le C \cdot \|\Pi_h u - u\|_{H^1} \cdot \|\Pi_h u - u_h\|_{H^1}$

Continuity of $a(\cdot,\cdot)$

$\Rightarrow \|\Pi_h u - u_h\|_{H^1} \le C \cdot \|\Pi_h u - u\|_{H^1}$.

c) $\|u-u_h\|_{H^1} = \|u - \Pi_h u + \Pi_h u - u_h\|_{H^1} \overset{\Delta}{\leq}$

$\leq \|u-\Pi_h u\|_{H^1} + \|\Pi_h u - u_h\|_{H^1} \leq$

$\overset{b)}{\leq} \hat{C} \|u-\Pi_h u\|_{H^1}.$

Old exam

Ex 6 from 26.08.21

6. Consider the domain $\Omega \subset \mathbb{R}^2$ in Figure 1. Set $A = (-1,0), B = (0,-1), C = (1,0), D = (0,1), E = (0,0), \Gamma_0 = \overline{CD}, \Gamma_1 = \partial\Omega \setminus \Gamma_0$. Let $V = \{v \in H^1(\Omega) : v = 0 \quad \text{on}\, \Gamma_0\}$ and consider the variational problem

$$\text{Find } u \in V \text{ such that } a(u,v) = \ell(v) \quad \text{for all } v \in V,$$

where the the bilinear and linear forms are defined by

$$a(u,v) = \int \nabla u(x,y) \cdot \nabla v(x,y)\, dx\, dy \quad \text{and} \quad \ell(v) = \int v(x,y)\, dx\, dy.$$

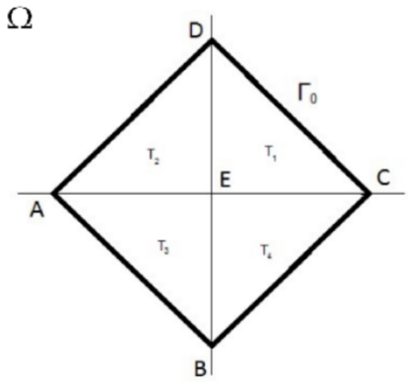


Figure 1: Courtesy from S.G. Rodriguez.

(a) Define the FE space V_h for an approximation of the solution to this variational problem by cG(1) FE on the grid defined in the figure. (2p)

(b) Provide a basis for the space V_h and write the FE solution u_h with help of the basis elements. (3p)

Hint: A piecewise linear continuous function on Ω is uniquely determined by its values at the nodes A, B, C, D, E. In this part of the question, you don't need to give an explicit formula for the basis elements, some precise explanatory text is enough.

(c) Give the final linear system $M\zeta = L$ (3 eq. for 3 unknown) coming from the FE problem corresponding to the above variational problem. (3p)

(d) Compute explicitely the first two diagonal terms in the matrix M. (3p)

Hint: Here you need explicit formulas for the basis elements. If you are not able to derive explicit formulas for the basis functions on the nodes A, B, E, you can use the following formulas

$$\varphi_A(x,y) = \begin{cases} 0 & \textit{for } (x,y) \in T_1 \cup T_4 \\ -x & \textit{for } (x,y) \in T_2 \cup T_3 \end{cases}$$

and

$$\varphi_B(x,y) = \begin{cases} 0 & \textit{for } (x,y) \in T_1 \cup T_2 \\ -y & \textit{for } (x,y) \in T_3 \cup T_4 \end{cases}$$

and

$$\varphi_E(x,y) = \begin{cases} -x-y+1 & \textit{for } (x,y) \in T_1 \\ x-y+1 & \textit{for } (x,y) \in T_2 \\ x+y+1 & \textit{for } (x,y) \in T_3 \\ -x+y+1 & \textit{for } (x,y) \in T_4. \end{cases}$$

Points will be deducted accordingly.

polyn. of degree ≤ 1

a) $V_h = \{ v \in \mathcal{C}^0(\bar{\Omega}) : v|_{T_i} \in P^{(1)}$ for $i = 1,2,3,4$ and $v = 0$ on $\Gamma_0 \}$

b) Here, we know that each $v_h \in V_h$ is determined by the values at the nodes A, B, C, D, E and obtained as a linear combination of hat fct $\varphi_A, \varphi_B, \varphi_C, \varphi_D, \varphi_E$.

But $v_h \equiv 0$ on Γ_0 for $v_h \in V_h$, hence $v_h(C) = v_h(D) = 0$

That is the basis for V_h consists of $\{\varphi_A, \varphi_B, \varphi_E\}$.

One has $u_h = u_h(A)\varphi_A + u_h(B)\varphi_B + u_h(E)\varphi_E$ $(*)$

c) The FE problem reads:

(FE) Find $u_h \in V_h$ s.t. $a(u_h, v_h) = \ell(v_h)$ $\forall v_h \in V_h$

From this, we find the linear system $M\xi = L$ by inserting $(*)$ in (FE) and taking $v_h = \varphi_A, \varphi_B, \varphi_E$

One then gets:

$$\sum_{j \in \{A,B,E\}} \overset{\text{unknown}}{\xi_j}\, a(\varphi_j, \varphi_i) = \ell(\varphi_i) \quad \text{for } i \in \{A, B, E\}$$

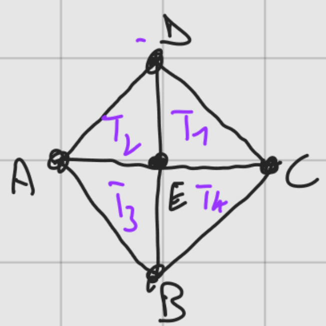
matrix M RHS L

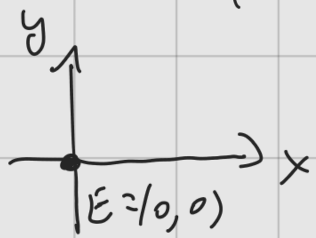
$$L = \begin{pmatrix} \int_\Omega \varphi_A \\ \int_\Omega \varphi_B \\ \int_\Omega \varphi_E \end{pmatrix} \quad (3\times 1) \quad \text{and} \quad \zeta = \begin{pmatrix} \zeta_A \\ \zeta_B \\ \zeta_E \end{pmatrix} \quad \text{and}$$

$$M = \begin{pmatrix} \int_\Omega |\nabla \varphi_A|^2 & \int_\Omega \nabla\varphi_A \cdot \nabla\varphi_B & \int_\Omega \nabla\varphi_A \nabla\varphi_E \\ * & \int_\Omega |\nabla\varphi_B|^2 & \int_\Omega \nabla\varphi_B \cdot \nabla\varphi_E \\ * & * & \int_\Omega |\nabla\varphi_E|^2 \end{pmatrix}$$

symmetric

d) We recall that $\varphi_A = \begin{cases} 1 & \text{on node } A \\ 0 & \text{on } B, C, D, E \end{cases}$





We know that $\varphi_A(x,y) = a + bx + cy$ on a triangle

In addition, we know that $\varphi_A = 0$ on all nodes of T_1 and T_4. Hence, $\varphi_A = 0$ on $T_1 \cup T_4$,

$$0 = \varphi_A(D) = b \cdot 0 + c \cdot 1 \quad \Rightarrow c = 0$$

On T_2, one has $0 = \varphi_A(E) = a + b \cdot 0 + c \cdot 0 \Rightarrow a = 0$

$$1 = \varphi_A(A) = b(-1) \quad \Rightarrow \quad b = -1$$

$\Rightarrow$ $\varphi_A(x,y) = -x$ on T_2. This is the same on T_3

That is, we get the expression

$$\varphi_A(x,y) = \begin{cases} 0 & \text{on } T_1 \cup T_4 \\ -x & \text{on } T_2 \cup T_3 \end{cases}$$

Finally, we compute the first element on diagonal of M:

$$M_{11} = \int_\Omega |\nabla \varphi_A(x,y)|^2 \, dxdy \overset{\text{Def } \varphi_A}{=} \int_{T_2 \cup T_3} |\nabla \varphi_A(x,y)|^2 \, dxdy \overset{\text{Def } \varphi_A}{=}$$

$$= \int_{T_2 \cup T_3} (-1,0)\begin{pmatrix} -1 \\ 0 \end{pmatrix} dxdy = \int_{T_2 \cup T_3} dxdy =$$

$$= \text{area}(T_2 \cup T_3) \underset{\substack{\text{Compute area of} \\ \triangle\, T_2 \cup T_3}}{=} \frac{1}{2} \cdot 2 \cdot 1 = 1 = M_{22}$$