Examination, 14 March 2022
TMA372 and MMG800

Read this before you start!
I'll try to come at ca. 09:45 (observe that you are in 4 different rooms).
Aid: Personal pocket calculator.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Use a proper pen and order your answers if possible.
The test has 3 pages and a total of 30 points.
Preliminary grading limits: 3:15-21p, 4:22-28p, 5:29p- (Chalmers) and G:15-26p, VG:27p- (GU)
Valid bonus points will be added to the total score if needed.
You will be informed via Canvas when the exams are corrected.
Good luck!

Some exercises were taken from, or inspired by, materials from *C. Cotter*, *M.J. Gander*, *F. Kwok*, *M.G. Larson*.

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1. Provide concise answers to the following short questions:
   (a) Which function lives in the trial space in the variational form of a PDE? (1 p)
   (b) Is the energy of the homogeneous linear wave equation with homogeneous Dirichlet BC a conserved quantity? (1 p)
   (c) Use one step of the trapezoidal rule to approximate the area under the function $f(x) = x^2$ between $x = 0$ and $x = 1$. (1 p)
   (d) What is an estimate for the error, measured in the energy norm, of the (continuous and piecewise linear) FEM with mesh size h for the model problem (Poisson's equation in $1d$ with homogeneous Dirichlet BC) seen in the lecture? (1 p)
   (e) Why is adaptivity useful? (1 p)
   (f) What is the point matrix in a FEM implementation? (1 p)

2. Show that the function $y(x) = \mathrm{e}^x$ is the unique solution to the initial value problem

$$\begin{cases} y'(x) = y(x) \quad \text{for} \quad x > 0 \\ y(0) = 1. \end{cases}$$

   Approximate the value of e^1 by applying two iterations of the explicit Euler's method with step size $h = \frac{1}{2}$. (2 p)

3. Deduce from the stability estimate

$$\|u(\cdot,t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \left\|f(\cdot,s)\right\|_{L^2} \, \mathrm{d}s$$

that the (classical) solution to the heat equation

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) \\ u(0,t) = 0 = u_x(1,t) \\ u(x,0) = u_0(x) \end{cases}$$

is unique. Here, $0 < x < 1$, $0 < t < T$, and f, u_0 are (nice and) given. (2 p)

4. Let $\Omega = (0,1)$ and $f \in L^2(\Omega)$. Consider the BVP

$$\begin{aligned} -u''(x) + u(x) = f(x) \quad &\text{for } x \in \Omega \\ u(0) = u(1) = 0. \end{aligned}$$

   (a) Write the variational/weak form of the above problem using a bilinear form denoted by $a(\cdot,\cdot)$. (2 p)

   (b) Let now $u_h \in V_h^0$ be the (cG(1)) Galerkin approximation of the solution u to the variational formulation from the previous point. That is

   $$a(u_h, v_h) = (f, v_h)_{L^2} \quad \text{for all } v_h \in V_h^0 \subset H_0^1(\Omega).$$

   Prove that u_h is the best approximation of u in V_h^0 in the H^1 norm. (3 p)

   (c) Which tool can you then use to derive an a priori error estimate for the Galerkin approximation u_h in the H^1-norm. You just need to name the tool/estimate in words, no need to provide an exact formula. (1 p)

5. Explain what the Lax–Milgram theorem says and what it can be used for. Be concise. (2 p)

6. Show that the bilinear form

$$a(u,v) = \int_0^1 \left(u'(x)v'(x) + (2+\cos(x))u(x)v(x)\right) \, \mathrm{d}x$$

is continuous and coercive in $H^1(0,1)$. (3 p)

7. Provide the nodal basis function $\Phi_1(x)$ for the finite element $\left(K = [0,1], P^{(2)}(K), \Sigma\right)$, where $P^{(2)}(K)$ denotes the set of polynomials of degree less or equal to two and $\Sigma = (L_1, L_2, L_3)$ is given by

$$L_1(f) = f'(0), \quad L_2(f) = f'(1), \quad L_3(f) = \int_0^1 f(x)\, \mathrm{d}x. \qquad \text{(2 p)}$$

Hint: Determine the constants a, b, c in $\Phi_1(x) = ax^2 + bx + c$.

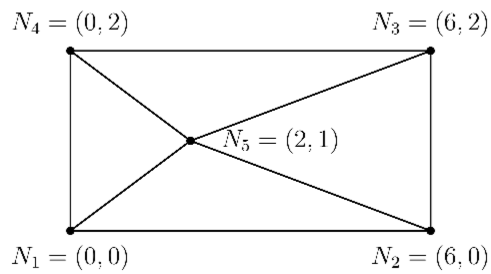

Figure 1: Courtesy from M.G. Larson.

8. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and consider the problem: Find $u = u(x, y)$ such that

$$\begin{cases} -\nabla \cdot (a\nabla u) + bu = f & \text{in } \Omega \\ -n \cdot (a\nabla u) = \gamma(u - g) & \text{on } \partial\Omega, \end{cases}$$

where $a = a(x, y) \geq \alpha > 0$, $b = b(x, y) \geq 0$, $f = f(x, y)$, $g = g(x, y)$, and $\gamma \geq 0$ are given. Here, $n = n(x, y)$ denotes the outward unit normal to Ω.

(a) Derive the variational formulation to the above problem. (1 p)
Hint: Remember Green's formula from the lecture

$$\int_\Omega \Delta u v \, dxdy = \int_{\partial\Omega} (n \cdot \nabla u) v \, ds - \int_\Omega \nabla u \cdot \nabla v \, dxdy.$$

(b) Provide the FE problem that comes from a $cG(1)$ discretisation of this variational formulation. (1 p)

(c) Derive the linear system of equations

$$(A + B + M)\,\zeta = \Gamma + R$$

resulting from the FE problem. Here, A, B, M are $(N_p + 1) \times (N_p + 1)$ matrices and Γ, R are $(N_p + 1) \times 1$ vectors, where one uses the basis functions $\{\varphi_j\}_{j=0}^{N_p}$ for the FE space. (2 p)

(d) Consider now the above problem with $a = 1, b = 0, \gamma = 0$ and $\Omega = [0, 6] \times [0, 2]$ with the triangulation from Figure 1. Compute the entry $A_{1,2}$ of the (global) stiffness matrix. (3 p)
Hint: The basis functions on a triangle are of the form $a_0 + a_1 x + a_2 y$ for some real numbers a_0, a_1, a_2. Observe that you can do this part of the exercise even if you have not done the previous parts.