Exam 14.03.22, TMA372/MMA800

① a) The sol to VF ①

b) Yes ①

c) $\int_0^1 x^2 dx \approx \frac{1}{2}(1^2+0^2) = \frac{1}{2}$ ①

d) $\|u-u_h\|_E \leq C \cdot h$ ①

e) To possibly save computational time or obtain an error at a given tolerance ①

f) The matrix that contains the coordinates of the nodes of a triangulation. ①

② $y'(x) = e^x = y(x)$ and $y(0) = e^0 = 1$ → sol IVP, unicity is standard result (Lipschitz cond.) 0.5

Euler: $y_1 = y_0 + h y_0 = 1 + \frac{1}{2} \cdot 1 = \frac{3}{2}$ 0.5 0.5

$y_2 = y_1 + h y_1 = \frac{3}{2} + \frac{1}{2} \cdot \frac{3}{2} = \frac{9}{4} \approx y(1)$ 0.5

③ Assume 2 sol, u_1 and u_2 ⇒ $u_1 - u_2$ solves $\begin{cases} u_t - u_{xx} = 0 \\ u(0,t) = 0 = u_x(1,t) \\ u(x,0) = 0 \end{cases}$ ①

Hence: $\|u_1 - u_2\|_{L^2} \leq \|0\|_{L^2} + \int_0^t \|0\|_{L^2} ds = 0 \Rightarrow u_1 = u_2$ ①

classical sol, $u_j \in C^2(0,1)$

④ a) Test with $v \in H_0^1(\Omega)$ and get ... ①

(VF) Find $u \in H_0^1(\Omega)$ s.t. $\underbrace{(u',v')_{L^2} + (u,v)_{L^2}}_{a(u,v)} = (f,v)$ $\forall v \in H_0^1(\Omega)$ ①

b) (VF): $a(u,v_h) = (f,v_h)_{L^2}$ and (FE): $a(u_h,v_h) = (f,v_h)_{L^2}$ → $a(u-u_h,v_h) = 0$ $\forall v_h \in V_h^0$ ①

Now, $\|u-u_h\|^2_{H^1(\Omega)} \underset{\text{Def } H^1}{=} a(u-u_h, u-u_h) \overset{②}{=} a(u-v_h, u-u_h) + \underbrace{a(v_h-u_h, u-u_h)}_{=0 \ (G.O)}$

$= a(u-v_h, u-u_h) \underset{C\text{-}S}{\leq} \|u-v_h\|_{H^1} \|u-u_h\|_{H^1} \Rightarrow \|u-u_h\|_{H^1} \leq \|u-v_h\|_{H^1}$ $\forall v_h \in V_h^0$ ①

c) Interpolation error estimates: $\|u - \pi_h u\| \leq \ldots$ ①

since $\pi_h u \in V_h^0$

5) Under some assumptions, LM says: $\exists!$ sol. u in some Hilbert space of the problem: find $u \in H$ s.t. $a(u,v) = l(v)$ $\forall v \in H$. 1

LM can be used to prove $\exists!$ sol. to linear (elliptic) BVP or PDE or FE problems coming from such problems/DE. 1

1) Continuity: $a(u,v) \underset{|\cos(x)| \le 1}{\overset{CS}{\le}} \|u'\|_{L^2} \|v'\|_{L^2} + 3\|u\|_{L^2} \|v\|_{L^2} \underset{\substack{\text{Def } H^1 \\ 1 \le 3}}{\le} 3\|u\|_{H^1} \|v\|_{H^1}$ 0.5 1

H^1-coercivity: $a(u,u) \overset{b}{\ge} \int_0^1 ((u'(x))^2 + u(x)^2)\,dx \underset{\text{Def } H^1\text{-norm}}{=} \|u\|_{H^1}^2$ 0.5 1

) $\Phi_1(x) \in P^{(2)}(K) \Rightarrow \Phi_1(x) = ax^2 + bx + c$ for some constant $a, b, c \in \mathbb{R}$.

We have the conditions:

$1 = L_1(\Phi_1) = \Phi_1'(0) = b \Rightarrow b = 1$

$0 = L_2(\Phi_1) = \Phi_1'(1) = 2a + 1 \Rightarrow a = -\frac{1}{2}$ 1 $\longrightarrow \Phi_1(x) = -\frac{1}{2}x^2 + x - \frac{1}{3}$ 1

$0 = L_3(\Phi_1) = \int_0^1 \Phi_1(x)\,dx = -\frac{1}{6} + \frac{1}{2} + c \Rightarrow c = -\frac{1}{3}$

)a) Test with $v \in H^1(\Omega)$:

$\int_\Omega f v\,dxdy = -\int_\Omega \nabla\cdot(a\nabla u) v\,dxdy + \int_\Omega buv\,dxdy \overset{\text{Green}}{=} \int_\Omega a\nabla u\cdot\nabla v\,dxdy - \int_{\partial\Omega} (n\cdot a\nabla u) v\,ds + \int_\Omega buv\,dxdy =$

$\underset{BC}{=} \int_\Omega a\nabla u\cdot\nabla v\,dxdy + \int_{\partial\Omega} \gamma(u - g) v\,ds + \int_\Omega buv\,dxdy$

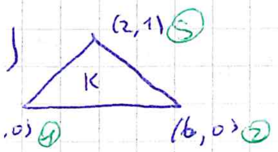
F reads: Find $u \in H^1$ s.t. $\int_\Omega a\nabla u\cdot\nabla v\,dxdy + \int_\Omega buv\,dxdy + \int_{\partial\Omega} \gamma uv\,ds = \int_\Omega fv\,dxdy + \int_{\partial\Omega} \gamma g v\,ds$ $\forall v \in H^1$ 1

) Find $u_h \in V_h$ s.t. $\int_\Omega a\nabla u_h \nabla v_h + \int_\Omega b u_h v_h + \int_{\partial\Omega} \gamma u_h v_h = \int_\Omega f v_h + \int_{\partial\Omega} \gamma g v_h$ $\forall v_h \in V_h$

where $V_h = \text{span}(\{\varphi_j\}_{j=0}^{N_h}) \subset H^1$, φ_j hat fct on a given triangulation 1

) Take $u_h = \sum_{j=0}^{N_h} \xi_j \varphi_j$ and $v_h = \varphi_i$ in FE to get $A = \left(\int_\Omega a\nabla\varphi_j\nabla\varphi_i\right)_{ij}$, $B = \left(\int_\Omega b\varphi_j\varphi_i\right)_{ij}$, etc 2

)



$A_{12} = \int_K \nabla\varphi_1\cdot\nabla\varphi_2\,dxdy \overset{\dots}{=} \dots = \frac{7}{36}\,\text{area}(K) = \frac{7}{36}\cdot\frac{6\cdot 1}{2} = \frac{7}{12}$ 1

$\varphi_1(x,y) = 1 - \frac{1}{6}x - \frac{2}{3}y$ 1 , $\varphi_2(x,y) = \frac{1}{6}x - \frac{1}{3}y$ 1